Teaching mathematical concepts: Instruction for abstraction

Michael Mitchelmore and Paul White
Macquarie University; Australian Catholic University National, Sidney Australia

ICME 10 2004
RL
Regular Lecture

Introduction

Everyone agrees that mathematics is an abstract subject. The abstractions to be learnt in mathematics consist largely of concepts and the relationships between them. The process of learning these abstractions - lso called abstraction - is therefore fundamental to the learning of mathematics.

Unfortunately, even though the word abstract is used a great deal, the process of abstraction is poorly understood. There are very few articles about it in the research or professional literature, and the topic seems sometimes to be studiously ignored. Such articles that are published are often concerned with points of view or fine distinctions, and serve to obscure the topic rather than illuminate it. This is surely an undesirable situation. If teachers could understand more clearly how their students learn the subject they are attempting to teach, teachers would be able to teach better and students would learn more.

The aim of this paper is to show how important abstraction is in elementary mathematics learning, and to suggest how to promote abstraction in the course of mathematics teaching. We will avoid technical terminology wherever possible.

Learning fundamental mathematical concepts

An example

As a start, let us look at an example. Six-year-old Elise is being interviewed as part of a research project into early number learning (Wright, Stanger, Cowper, and Dyson, 1996).


Elise being interviewed by her teacher

The teacher follows a standard procedure: She puts a number of counters on the table, covers them with a piece of card, puts some more counters next to it, and asks the child how many counters there are altogether. The aim is to find out if the child understands the conceptual basis of addition.

On 15 October, this is how the interview proceeds. Elise has no difficulty when there are 5 counters under the card and 3 next to it:

T: 5 here …and 3 more here.
E: (*counts in mid-air, pointing to imagined counters*) 1, 2, 3, 4, 5, 6, 7, 8.
T: Count them.
E: (*uncovers the counters*) 1, 2, 3, 4, 5, 6, 7, 8.
T: Good girl.

However, she has far more trouble with 10 counters under the card:

T: What about if I put 10 here and 3 here.
E. (*again counts in mid-air*) 1, 2, 3, 4, 5, 6, 7, 8, 9 (*loses count*) 10, … 11, …12?
T: Have a think, have another think, think carefully. Start with this side. How many under the card?
E: 10
T: Right, start with this side.
E: (*presses on counters through the card*) 1, 2, 3, 4, 5 …
T: No, don't feel them. How many are here?
E: 10 there.
T: All right, and 3 here.
E: 3 there. 1, 2, 3, 4, 5, 6, 7 … (*loses count*)
T: Stop, 10 here.
E: 10, 3.
T: Start at 10 and count these ones on.
E: (again *presses on counters through the card*) 1, 2, 3, 4 …
T: No, no, don't touch them. We know how many are there. How many are there? (*removes card*) You count them.
E: 1, 2, 3, 4, 5 … (*rearranges counters in pairs*) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
T: That's right, 10. Now do we have to count them again? How many are there?
E: 10
T: Now let's just say 10.
E: 3
T: Now count three more. 10?
E: 3
T: Yes. How many altogether?
E: (*counts in mid-air, pointing to paired counters under card*) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13?
T: (*uncovers the counters*) Show me.
E: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13!

It is clear that, despite the teacher's best efforts to guide her, Elise cannot "count on". She can only complete this type of task successfully when she can visualise the counters under the card. (Pictorial stage) Even when she has counted the objects herself, she makes no further use of the number. The question "How many?" appears to be a command to follow a standard counting procedure and emphasise the last numeral reached, nothing else. It is fair to say that Elise had a very limited concept of number.

Elise was interviewed again on 5 November. This time, she counts on with confidence:

> T: (*puts 12 counters under the card*) 12, and I've got 4 more here.
> E: 12, … (*counts remaining 4 counters in mid-air*) 16!
> T: Yeh.

Sometime in the intervening 3 weeks, Elise has learnt to count on. The answer to the question "How many?" has become a property of the set of objects counted, something that has a meaning and can be used in further operations. In other words, there has been a fundamental change in her understanding. A new mental object has been created, which she can manipulate as easily as the counters themselves. It is fair to call this mental object a *concept.* By 5 November, Elise has such a clear concept of number that, as later stages of the interview show, she can even use it with "pretend" numbers where there are no counters under the card at all. "Number" has already developed an independent existence in her mind.

Although Elise had probably been counting for several years before these interviews, the change in Elise's understanding was relatively sudden. It was also irreversible - Elise would now never go back to counting the counters once she knows how many there are. The very irreversibility makes it difficult for us (and her, and her teacher) to understand how Elise thought about numbers before this stage.

The obvious question arises, how did Elise learn to count? We think it was by recognising the common features, the similarities, the patterns in all the counting tasks she had engaged in. She had noticed that, whatever objects she counted and however many of them there were, whenever she re-counted the objects she reached the same number in the counting sequence. How else could she have learned to count on? The 15 October interview transcript shows that, no matter how many times the teacher suggested a quicker method of solving the task, she simply could not tell Elise how to count on. The teacher's suggestions simply did not make any sense to Elise at the time. But at some time in the following three weeks, Elise finally recognised the pattern the teacher was referring to. Elise's learning was entirely a result of her own efforts to make sense of the situation, and not the result of the teacher telling her what to do.

Concepts are learned by empirical abstraction

The hypothetical process by which Elisa learnt the concept of number is called *empirical abstraction*. It occurs when a learner recognises underlying similarities between superficially different experiences. As a result, a new mental object, a concept, is formed which in a sense embodies this similarity. We say that the concept *reifies* (makes an object out of) the similarity. Skemp (1986) described the process like this:

> "*Abstracting* is an activity by which we become aware of similarities ... among our experiences. *Classifying* means collecting together our experiences on the basis of these similarities. An *abstraction* is some kind of lasting change, the result of abstracting, which enables us to recognise new experiences as having the similarities of an already formed class. ... To

> distinguish between abstracting as an activity and abstraction as its end-product, we shall ... call the latter a *concept.*" (p. 21, italics in original)

Empirical abstraction is not, of course, restricted to the concept of number. Most everyday concepts (like colour, friend, and fairness) are the result of the same process, as children and adults recognise underlying similarities between ostensibly different objects, situations or circumstances.

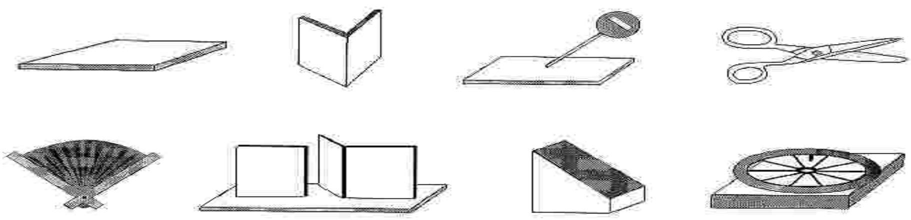
Many writers claim that empirical abstraction is not mathematical. Piaget, for example, distinguished the empirical abstraction involved in everyday concept formation from what he called *reflective abstraction* in the formation of logical-mathematical concepts. We claim that all fundamental mathematical ideas are learnt through a common process of empirical abstraction, and that further learning builds on these empirical concepts by a different process called mathematical abstraction. To back up this claim, we obviously need to look at some more examples.

Circles

Round objects are very common in children's experience. They recognise roundness very early in life. (Michael vividly remembers my two-year-old daughter walking around a circle of stones in a modern art exhibition chanting "Round and round, round and round".) Then they recognise that some round objects are "perfectly round" and call them circles. Our interpretation of this process is that children have recognised a similarity between a number of perfectly round objects and reified it into the concept of circle. This concept then becomes for children a mental object that they can manipulate like any other: They can identify circles around them, talk about circles without any round objects being present, and recognise and represent circles graphically.

Angles

Adults can recognise angles in many superficially different contexts. Here are diagrams of models we have used to depict a variety of angle contexts:



These contexts vary widely. As can be seen from the models, an angle may be formed by two lines, two planes, a line and a plane, two hinged lines, a single rotating line, a single sloping line, or a number of rotating lines. The link between all these contexts is obscure, to say the least, and it should not be surprising therefore that students find the angle concept difficult (Mitchelmore and White, 2000).

But there is an underlying similarity between all the above contexts: Each one can be interpreted *as if* it consisted of two lines meeting at a point, and in each case the way the two lines meet has a particular significance (different in each context). In a sense, the concept of angle *is* this similarity. As before, empirical abstraction takes a similarity and reifies it into a mental object.

It is not possible to give a full definition of the concept of angle without restricting it to one context (as in “an amount of turning”) or getting into a circular definition (as in “the angular relation between two lines”) - therefore it is impossible to teach it. The only way students can learn the angle concept is through their own efforts, that is, by empirical abstraction from their experience. That is not, of course, to deny that a teacher’s guidance might make it easier for them to make this abstraction, just as Elise’s teacher’s suggestions helped her recognise a counting pattern.

Area

Area is rather like angle - it is impossible to define without being restrictive or circular. For example, “the amount of space in a region” does not define area if you do not already have a concept of “amount of space”. And this concept could only come from experience with regions and tasks that somehow depend on their area. Such tasks would include the following:

- Table tops - different sizes allow more or less space for activities such as playing with toys
- Paper - more/less space for scribbling, painting or cutting out
- Floor, garden - more/less space to move around in/make a mess in
- Boxes, trolleys, trays - more space to fill with blocks etc.
- Reading books - space for words/pictures/stories depends on area of page
- Writing books - space to write, paste
- Plates - space for food
- Bed, bedding - space to fit and move around
- Pots and pans - space to boil, bake or fry food

As children see that all these various tasks “essentially” involve the same thing, that “thing” develops into the concept of area. Once again, a fundamental mathematical concept arises through empirical abstraction from children’s experience.

Rate of change

Not all fundamental mathematical concepts are as elementary as number, circle, angle, and area. For example, the topic of rate of change is usually taught in secondary schools, but in order to be meaningful it must be related to examples which students have experienced or at least read and thought about. There is, in fact, an enormous number of situations where one variable depends on another and the rate of change is of significance (e.g., population growth, cost of living, radioactive decay, cooling). A definition can be given, but it will only be meaningful if students can see that it applies to a variety of such situations. In other words, they should already have a concept of rate of change before a mathematical definition can make sense. And this concept can only be learned through empirical abstraction, by recognising what is similar across all rate-of-change situations.

There are many more examples we could give. Just briefly, here are some more:

- *Multiplication* (reifying the similarity between such situations as equivalent groups, equivalent measures, comparisons, multiplicative changes, arrays, area, combinations, etc.)

- *Fraction* (reifying the similarly between partitioning, sharing, scaling, comparison, measurement, rates, and relative size)
- *Congruence*, similarity (similar to the concept of circle)
- *Length, volume* (similar to area)

We hope we have given enough examples to make our first main point:

> *Conclusion 1: All fundamental mathematical concepts are initially learned through empirical abstraction.*

Later mathematics learning

Later learning - formal mathematics

Empirical concepts are vague and cannot be easily defined. For example, you can only show what round means by producing some examples - and there will always be some borderline cases. Mathematics *formalises* empirical concepts into precise mathematical concepts. For example, when a circle is defined as the set of points in a plane equidistant from a fixed point, there is no ambiguity - provided, of course, you know what the other formal terms (set, points, plane, equidistant, fixed) mean. Take any of the concepts we discussed in the last section, and you will see that it can be formalised by stating quite precisely what it means in terms of other formal notions.

The purpose of formalisation is to build up a system that can be easily manipulated independently of experience. For example, many 6-year-old children can find how much money they need in order to buy \$2.35 worth of sugar and \$4.45 worth of coffee by "playing" with notes and coins. Then they learn some arithmetic, and a couple of years later they can do such calculations without the play with physical money. Here is another example: A weight-watcher wanted to adjust a recipe by working out two-thirds of three-quarters of a block of butter. To do this, he took a block of butter, cut off one-quarter, and then cut off one-third of the remainder. How much simpler had he just calculated $\frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$ before cutting!

This possibility of predicting without doing is one reason why mathematics is so powerful. Another reason is that, ultimately, almost all mathematical ideas can be traced back to empirical ideas. Mathematics does not simply spring into mathematicians' brains from nowhere, but is either suggested by empirical problems or by other mathematical notions that owe their origin to empirical problems.

Links between mathematical ideas and empirical concepts

We claim that students' formal mathematical ideas are of two types (in the extremes - of course intermediate cases exist). These are:

- An a*bstract-general* idea is based on a known empirical concept. It formalises the general pattern or relationship which is embodied in that concept. The relation to an empirical concept gives an abstract mathematical idea *meaning* and shows the *purpose* of developing procedures for manipulating it.
- An *abstract-apart* idea is unrelated to any empirical concept, but is defined solely in relation to other formal ideas. If these other ideas have themselves no meaning or purpose, then the new idea will have no meaning or purpose.

Many students only have an abstract-apart understanding of fundamental mathematical ideas such as angles, fractions, and rate of change. Here are some examples that we have encountered in our experience as teachers and researchers:

- Julie can calculate angles on a straight line, but she cannot sketch angles of a given size
- Jose can work out the angle sum of a polygon, but cannot explain what an angle is
- Carl gives a correct proof of the centre-chord theorem for circles, but does not see any relation to symmetry
- Briony correctly completes the ratio 5:3 = 12:?; but when asked to enlarge a 5 by 3 rectangle so that its length becomes 12, says that the width would be 10
- Ahmed can differentiate an algebraic function, but cannot find realistic rates of change
- Maria can maximise *y* as a function of *x*, but not *A* as a function of *r*
- Students leave glaring errors (e.g., 4.2 busses)
- Students work by superficial analogy (e.g. $\frac{1}{2}+\frac{2}{3}=\frac{3}{5}$, $(x + y)^2 = x^2 + y^2$).
- Students cannot give examples of mathematical ideas

For the students who make such errors, mathematics is mysterious, boring, quickly forgotten nonsense. Because the ideas are meaningless, they have no way to check the correctness of their answers and can only rely on the authority of the teacher or the textbook.

There are two reasons why this situation may arise: Either the students do not link a mathematical idea to an empirical concept, or they have not formed the relevant empirical concept in the first place. This is our second main point:

> *Later mathematics learning needs to be built on a firm foundation of empirical concepts.*

How can teaching encourage empirical abstraction?

Current teaching methods do not promote empirical abstraction

Traditional teaching provides definitions on the basis of very few examples, sometimes none. But, as we have argued above, a definition cannot define unless the concept already exists. For example, a Year 8 textbook we saw recently explains that "a ratio is a comparison of like quantities". This statement is meaningless unless you have already made quantitative comparisons and know when it is appropriate to compute ratios (instead of differences, for example). No wonder students learn to compute with ratios but can never apply them.

Traditional mathematics teaching holds that it is important for students to learn techniques before applications. We called this the ABC method - *A*bstract *B*efore *C*oncrete (Mitchelmore, 1999). Techniques are initially taught in a context-free mode, the theory being that "knowledge acquired in 'context-free' circumstances is supposed to be available for general application in all contexts" (Lave, 1988, p. 9). However, because students have no chance of relating abstract mathematical ideas to concrete experience and therefore to empirical concepts, ABC teaching leads to *abstract-apart* ideas. Even if teachers reach the "problems" section at the end of the chapter, it might be too late to remedy the damage. This assertion is supported by Pesek and Kirshner (2001) in their study of the teaching of area and perimeter in Grade 5: They found that students taught to use the formulae for five lessons followed by three lessons on the conceptual basis of the formulae performed less well on a criterion test than students who were only taught the conceptual basis for three lessons.

In the 1960s, the *New Maths* failed for similar reasons. Mathematicians claimed that if young students were taught the abstract structure of various mathematical systems, they would be better able to compute and apply the mathematics they were learning. This again meant that students had no chance of relating abstract mathematical ideas to concrete experience until it was too late.

Even among progressive educational movements, teaching rarely focuses on building up empirical concepts. *Constructivism* has been the predominant paradigm in mathematics education circles for two decades now (Fitzpatrick, 1987). In this view of learning, individuals construct knowledge by making sense out of life's experiences (including mathematics) rather than having knowledge transmitted to them by more expert others. In fact, empirical abstraction is essentially a constructivist concept, but constructivists rarely talk about abstraction. Exceptions may be found in the works of Dubinsky (e.g., 1991) and von Glasersfeld (e.g., 1995).

Many elements of constructivist teaching could be expected to promote abstraction: The emphasis on discussion of existing knowledge or experience; the challenge of problem solving; the admission of contrasting methods and the reconciliation of conflicting solutions; the use of small group cooperative learning - all promote reflection which could lead to the recognition of similarities and hence the abstraction of new concepts. In particular, *activity methods* encourage hands-on exploration of concrete materials - but the purpose of these activities, in terms of what students should learn from them, is rarely explained and there is never any mention of how students might abstract concepts or relationships from the activities. Similarly, *connected mathematics* movements encourage students to look for links, which is of course very important, but they do not mention the possibility of using links between different contexts to identify similarities which could then be reified into an empirical concept.

Dienes (1963) developed a method that really did attempt to teach abstraction, by leading students to identify similarities between isomorphic structures. For example, place value concepts were abstracted from commonalities between his Multibase Arithmetic Blocks and tree diagrams. One of Dienes' basic principles was that of *perceptual variability*: "To abstract a mathematical structure effectively, one must meet it in a number of different situations to perceive its purely structural properties" (p. 158). Dienes aimed to create a teaching environment in which children learned by reflecting on their experiences in a variety of different situations. The theory of abstraction and generalisation underlying his teaching method (Dienes, 1961) looks very much like the empirical abstraction we described in the first section of this paper.

However, Dienes was not happy with the outcomes of his experiments. "We assumed ... that abstraction would arise from a multiple embodiment of the concepts to be abstracted. By this I mean that situations physically equivalent to the concept-structure to be learned would, if handled according to specific instructions leading towards the structure, result in abstracting the common structure from all the physical situations. ... But as we observed children going through the 'abstraction exercises', it soon became clear that the picture was far more complex than we had assumed" (1963, p. 68). For example, requests to do "something like this" in a different embodiment or to say "how they are alike" initially brought out common features instead of a common structure - but only the first time through, as if children were just not used to looking for deeper similarities. Nevertheless, he claims that "artificial exercises in forming isomorphisms could act as teaching devices to help [children] recognise similarities when they see them" (p. 85).

Our view of Dienes' experiments is that the various "concrete materials" he used to embody mathematical structures were only concrete in the sense of being constructed out of physical materials. They were not familiar objects in children's experience and were in fact already abstractions from that experience - abstractions made by the researcher and not by the children. This view is borne out by findings that children often have difficulty relating Dienes' blocks to arithmetical procedures (Boulton-Lewis, 1992). In Dienes' method, children are not making empirical abstractions that could provide meaning and purpose to their learning.

So how can you teach for empirical abstraction?

To see how it might be possible to teach for abstraction, let us look at a particular concept. We have chosen ratio, a topic frequently taught to students in Grades 7-8. To learn this topic meaningfully, students would need first to explore a variety of ratio contexts and then to abstract and reify the similarity between them. This would involve the following stages:

- *Explore a variety of contexts where ratios are significant and familiarise themselves with the mathematics within each context.* A large number of contexts are familiar to students from their own personal activities (e.g., mixtures, shape, gender balance) and they would have encountered many other contexts in adult affairs (odds, share price changes, voting swings, …). But it should not be assumed that students understand the mathematical structure of these contexts. For example, it is well known that many students of this age believe that the taste of a cordial mixture stays the same if you add equal quantities of cordial and water. Students need to explore and explain before they accept the need to increase the cordial and water by the same factor. Each ratio context has its own unique explanation as to why it scales up and down using multiplication and division and not, for example, addition and subtraction.
- *Identify similarities between the mathematics in these ratio contexts.* Students can be led to recognise what all the contexts explored have in common. They all involve a kind of "balance" between two like quantities, and this balance is maintained when both quantities are multiplied or divided by the same number. This particular kind of balance relationship may then be defined as a ratio.
- *Practise the procedures for working with ratios*. Because of the mathematical similarity between all ratio contexts, the same calculations are involved no matter what the numbers represent. These calculations can now be practised apart from any particular context in order to develop computational fluency.
- *Apply skills to problem solving*. Once students have learnt to compute fluently with ratios, they are now ready to do three things: (1) Revisit the original contexts and see how much more easily they can now solve problems in those contexts. (2) Investigate new contexts to see if they involve ratios. (3) Solve new problems in new contexts.

A similar process could be applied to the teaching of any fundamental mathematical idea which is built on empirical abstraction. We call it *Instruction for Abstraction* (Mitchelmore and White, 2000). It consists essentially of four steps:

1. *Familiarisation.* Students become familiar with several relevant contexts from which the concept will be abstracted before learning about the concept itself. These contexts may be objects (e.g., tiles, furniture, drawings), operations (e.g., mixing, sharing, filling, ...), or abstract ideas (e.g., price, balance, percentage, ...). Each example is discussed using the natural language peculiar to that context (e.g., taste, shape, multiple), not that of the concept to be abstracted (e.g., ratio). However, the teacher will anticipate the abstraction to be made later (e.g., by including examples of situations involving ratios of more than two quantities).
2. *Similarity recognition.* The concept is taught by making explicit the similarities underlying familiar examples of that concept. The similarities may be superficial (e.g., between the appearance of different round objects) or structural (e.g., between turning and sloping). Students' attention is directed to the critical attributes that define these similarities and that are embodied in the concept to be abstracted (e.g., the uniformity assumption underlying the area concept). The teacher then introduces the specialist language associated with the concept and uses this vocabulary to "define" the concept (in the sense of making it more definite) by showing how it relates to the similarities on which it is based
3. *Reification.* As students explore the concept in more detail, it becomes increasingly a mental object in its own right, detached from any specific context. Almost any use of the concept is likely to assist its reification, providing the relation between the abstract concept and familiar examples of the concept is maintained. Some possibilities:
   - Find how to use the concept in practice (e.g., by estimating the area of the school on a map).
   - Investigate how to operate on the abstract concept, but always relate the results to some familiar context (e.g., predicting the size of the angle formed by combining 30°and 60° angles, and then checking the prediction using tiles).
   - Define and work with special cases (e.g., rate of change as a ratio).
   - Look for generalisations involving the concept (e.g., area formulae).
4. *Application*: Students use what they have learned to solve old and new problems more efficiently. Returning to the contexts that were used to introduce the concept shows the power of the computational techniques they have learned. Solving problems in new contexts shows the value of the more general understanding they have achieved as well as generalising the concept to further contexts.

Instruction for Abstraction clearly has much in common with other constructivist approaches, and many of their principles apply equally well. One difference is that *Instruction for Abstraction* has no problem with the fact that much of the content of school mathematics is pre-determined. Our belief is that, instead of merely hoping that abstract mathematical ideas will develop as a result of cooperative learning, reflection on experience, and so on, a more deliberate attempt to foster the abstraction of crucial mathematical concepts would pay handsome dividends in terms of student learning and understanding.

Teaching the angle concept by Instruction for Abstraction

Instead of merely talking about how we think concepts should be taught, we now give an actual example of how it was done. In 2001, we were asked by the New South Wales Department of Education and Training to put into effect a hypothetical sequence for the teaching of angles in primary school by *Instruction for Abstraction*, which we had outlined in Mitchelmore and White (2000). We did this in field trials over two years (White and Mitchelmore, 2003).

Our research had suggested that even Year 2 children could learn to identify 2-line angles (i.e., angles in contexts with two visible arms) but that generalisation to angles with one or no visible arms would be difficult. So we designed a 10-lesson unit for Year 3 intending to teach 2-line angles and to see how well children could generalise to a few angle contexts in which only one arm was visible.

The first three lessons focussed on corners, including the corners of pattern blocks, corners in the room, and measurement of the size of a corner using a primitive "angle tester" (a paper protractor consisting of six lines through a point intersecting at 30°). Students investigated how scissors moved in Lesson 4, and in Lesson 5 they investigated other scissors-like objects. In all lessons, students matched angles in different contexts by superimposing one angle on the other. It was hoped they would be able to recognise that all the examples of angles involved (a) two lines, (b) a point where the lines meet, and (c) an amount of opening between the lines.

Lessons 6-8 each introduced one 1-line angle context: the hour hand of a clock, a door, and a sloping ruler. In each lesson, students firstly studied how the object moved and the significance of this movement. (For example, the hour hand of a clock moves from 2 o'clock to 5 o'clock in 3 hours.) They then investigated how to describe the size of such movements (e.g., by matching the 3-hour movement to a corner of a square pattern block), thereby linking back to the angles developed in earlier lessons. The learning activities were designed to help children identify the second, missing line of the angle in each context. Lesson 9 was an attempt to highlight the similarities between all the angle contexts studied in the unit, and Lesson 10 was an open-ended, creative activity designed to generalise the angle concept to other curriculum areas besides mathematics.

It may be seen that this unit incorporated the four principles of *Instruction for Abstraction*. Firstly, each context (pattern blocks, angles in the room, scissors, clocks, doors, and slopes) was investigated in its own right in order to increase students' *familiarity* with it - especially as regards possibly hidden features common to all angles. Secondly, the *similarity* between each new context and previous angle contexts was continuously stressed. Thirdly, *reification* occurred as students measure and compare angles in different contexts. Fourthly, students *applied* their understanding to other curriculum areas.

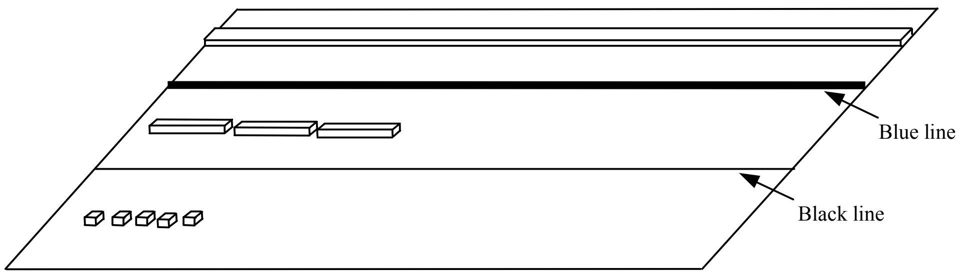
A total of 12 teachers from 5 schools taught the experimental unit. The teachers and their students reported that they liked the approach. In particular, they found the "angle tester" and the use of superimposition useful in linking angles in different contexts. All students made substantial gains on a set of similarity recognition tasks given (by the teachers) before and after teaching the unit. It appeared that, by the end of the unit, almost all the students had mastered the idea of a 2-line angle. By contrast, only just over a half of the students had generalised this idea to 1-line angles.

Following this feedback, the teaching sequence was split into two units, one for Year 3 and one for Year 4. The Year 3 unit included only material on 2-line angles, while the Year 4 unit revised the Year 3 work and then explored 1-line angles. The second trial involved 25 teachers from 7 schools. Once again, students (and teachers) showed considerable gains in understanding of the angle concept as a result of studying (or teaching) the unit. Several minor improvements were made, and the unit has now been published and distributed to all public schools in the state (New South Wales Department of Education and Training, 2003).

Teaching decimals by Instruction for Abstraction

We have also been able to test the validity of teaching of decimals by *Instruction for Abstraction* in a brief exploratory investigation which took place in a single classroom containing a mixture of Year 3 and Year 4 students. We identified three contexts where we thought decimals concepts would arise in students' experience: (1) money - dollars and cents; (2) measurement - metres and centimetres; and (3) 10 ´ 10 chocolate bars - invented to make children's experience with the hundreds square and Dienes blocks more "relevant". Our aim was to help children abstract the common idea of a 2-place decimal from these three contexts.

To act as a linking aid, we used a simplified version of the *Linear Arithmetic Blocks* (LAB) (Stacey, Helme, Archer, & Condon, 2001). In this model, students represented hundredths by a Dienes "short", tenths by a Dienes "long", and units (when needed) by metre rules. They could lay these objects out end-to-end to represent the size of a number by a length. Alternatively, they could lay them out on an "organiser" consisting of a 1-metre long strip of paper fixed to the students' desk, divided into three columns by a broad blue line and a thin black line. This is how children were taught to represent 1.35 (in each of the three contexts) using this organiser:



Following the standard *Instruction for Abstraction* procedure, students were first given activities to familiarise themselves with each context. They were then given challenges to add numbers in that context and to multiply by 10, firstly using their intuitive understanding and then using the LAB model. The meaning of each digit was also emphasised. All exercises were embedded in familiar situations such as spending pocket money, comparing heights, and sharing chocolates. After this introduction, students' attention was drawn to the similarities between the three contexts, both in terms of the superficial features (e.g., three digits) and structural characteristics (e.g., the methods for adding and multiplying by 10). A standard place value chart was then introduced as an abbreviation of the LAB model. The unit concluded with some abstract exercises which students were expected to complete either by interpreting them in any one of the three contexts or by using their abstract understanding of 2-place decimals. Unfortunately, there was no time to attempt application to other contexts.

A target group of 8 Year 3 students was observed in each lesson, and a short posttest was administered to these 8 students and to 8 students in a comparable class in the same school. The unit appeared to have been effective. For example, by the end of the unit 5 out of the 8 target students successfully calculated 0.34 + 0.7, which the teacher indicated was more than she would have expected of the Year 4 students under normal conditions.

Other examples of Instruction for Abstraction

We are currently (October 2004) engaged in a project to test the teaching of percentages by *Instruction for Abstraction*, and have plans to extend this to further multiplicative relationships next year.

But perhaps the best known example of *Instruction for Abstraction* (although it is not called this) is the highly successful *Realistic Mathematics Education* (RME) movement. This movement has been developing over the past 20 years, and the majority of Dutch primary and middle schools now follow its curriculum. It has also been adapted for the USA as *Mathematics in Context* (see http://mic.britannica.com/mic/common/home.asp). Treffers (1992, p. 26) describes the five basic principles of RME as follows:

1. Mathematics learning is constructive.
2. Learning proceeds over several levels over a long time period.
3. Reflection plays an important role in learning.
4. Learning is interactive.
5. Mathematical ideas are interconnected.

Typically, the RME approach to teaching a topic consists of three stages:

1. Develop rules of operation in several specific, familiar, everyday contexts.
2. Demonstrate that the same structure is present in several such contexts.
3. Formulate, symbolise and study the common structure.

Treffers (1992, p. 32) calls the first of these steps "the modeling of problem situations [so] that these can be approached with mathematical means." The second step consists of the recognition of structural similarities and the third step the construction of a new mental object, the two steps together being "directed at the perceived building and expansion of knowledge within the subject system, the world of symbols." We can recognise these three steps as almost identical to the first three steps of *Instruction for Abstraction*. The main difference is a greater emphasis on symbolisation than on abstraction. In fact, the *Realistic Mathematics Education* movement has only recently began to use the term abstraction. They prefer to use the term *emergent modeling* in which "formal mathematics grows out of the mathematical activity of the children" and refer to "abstraction as the creation of new mathematical reality" (Gravemeier, 2002, pp. 125).

Hence our final conclusion to this paper:

Mathematics teaching can and should put more emphasis on empirical abstraction.

Epilogue: Empirical abstraction is not everything

Despite our enthusiasm for empirical abstraction, we recognise that it is not the only way in which new mathematics is learned. Empirical abstraction only comes into play when students learn fundamental concepts such as those discussed above, upon which early formal mathematics is built. Later ideas are learned by a process that might be called *mathematical abstraction* (Mitchelmore & White, 2004). Here are some examples:

- Rational numbers (e.g., $\frac{1}{2}$ and $\frac{3}{4}$ are empirical, but $\frac{4}{7}$ and $-\frac{5}{3}$ are not)
- Powers (e.g., 2^2 and 2^3 are empirical [think of area and volume], but 2^4, 3^0, 2^{-1}, and $2^{1/2}$ are not)
- Irrational numbers (e.g., $\sqrt{2}$ does not exist empirically, because it cannot be calculated)

In mathematical abstraction, objects are invented in order to extend empirical rules or properties to cover cases that do not normally arise empirically, and these objects are then defined in such a way as to maintain consistency. But it is still true that mathematical ideas learnt in this way can be abstract-general (linked to known concepts through a meaningful definition) or abstract-apart (defined abstractly without any explanation).

Another very important aspect of mathematics learning is *generalisation.* As for abstraction, generalisations can be empirical (based on experience) or mathematical (based on definitions and properties of mathematical objects). Generalisation is clearly closely related to abstraction (Mitchelmore, 1999, 2002; White & Mitchelmore, 1999).

References

Boulton-Lewis, G. (1992). The processing loads of young children's and teachers' representations of place value and implications for teaching. *Mathematics Education Research Journal*, **4**(1), 1-23.

Dienes, Z. P. (1961). On abstraction and generalization. *Harvard Educational Review,* **31**, 281-301.

Dienes, Z. P. (1963). *An experimental study of mathematics learning.* London, England: Hutchinson.

Dubinsky, E. (1991). Constructive aspects of reflective abstraction in advanced mathematics. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 160-202). Dordrecht, The Netherlands: Kluwer.

Fitzpatrick, J. (1987). What constructivism might be in mathematics education. In In J. C. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the 11th annual conference of the International Group for the Psychology of Mathematics Education,* (Vol. 1, pp. 3-27). Montreal, Canada: Program Committee.

Gravemeier, K. (2002). Building new mathematical reality, or how emergent modeling may foster abstraction. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 125-129). Norwich, England: Program Committee.

Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life.* Cambridge, England: Cambridge University Press.

Mitchelmore, M. C. (1999). How to teach generalisations in mathematics? In J. M. Truran & K. M. Truran (Eds.), *Making the difference* (Proceedings of the 22nd annual conference of the Mathematics Education Research Group of Australasia, pp. 373-379). Sydney: MERGA.

Mitchelmore, M. C. (2002). The role of abstraction and generalisation in the development of mathematical knowledge. In D. Edge & Y. B. Har (Eds.). *Mathematics education for a knowledge-based era* (Proceedings of the Second East Asia Regional Conference on Mathematics Education and the Ninth Southeast Asian Conference on Mathematics Education, Vol. 1, pp. 157-167). Singapore: Association of Mathematics Educators.

Mitchelmore, M. C., & White, P. (2000). Teaching for abstraction: Reconstructing constructivism. In J. Bana & A. Chapman (Eds.), *Mathematics education beyond 2000* (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia, pp. 432-439). Perth: MERGA.

Mitchelmore, M. C., & White, P. (2000). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics,* **41**, 209-238.

Mitchelmore, M. C., & White, P. (2004). Abstraction in mathematics and mathematics learning. In In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 329-336). Bergen, Norway: Program Committee.

New South Wales Department of Education and Training. (2003). *Teaching about angles: Stage 2.* Sydney: Author.

Pesek, D. D., and Kirshner, D. (2001). Interference of instrumental instruction in subsequent relational learning. *Journal for Research in Mathematics Education,* **31**, 524-540.

Piaget, J., & collaborators. (1977). *Recherches sur l'abstraction réfléchissante* [Research on reflective abstraction] (2 vols.). Paris: Presses Universitaires de France.

Skemp, R. (1986). *The psychology of learning mathematics* (2nd ed.). Harmondsworth: Penguin.

Stacey, K., Helme, S., Archer, S., & Condon, C. (2001). The effect of epistemic fidelity and accessibility on teaching with physical materials: A comparison of two models for teaching decimal numeration. *Educational Studies in Mathematics*, **47**, 199-221.

Treffers, A. (1992). Didactical background of a mathematics program for primary education. In L. Streefland (Ed.), *Realistic mathematics education in primary school* (pp. 21-56). Utrecht, The Netherlands: CD-ß Press.

von Glasersfeld, E. (1995). Sensory experience, abstraction, and teaching. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 369-383). Hillsdale, NJ: Lawrence Erlbaum.

White, P., & Mitchelmore, M. C. (1999, November). *Learning mathematics: A new look at generalisation and abstraction* (Refereed proceedings of the combined conference of the Australian and New Zealand Associations for Research in Education, Melbourne). Available at http://www.swin.edu.au/aare/99pap/whi99309.htm

White, P., & Mitchelmore, M. C. (2003). Teaching angles by abstraction from physical activities with concrete materials. In Pateman, N. A., Dougherty, B. J., & Zilliox, J. T. (Eds.), *Proceedings of the 27th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 403-410). Hawaii: Program Committee.

Wright, R. J., Stanger, G., Cowper, M. & Dyson, R. (1996). First-graders' progress in an experimental mathematics recovery program. In J. Mulligan & M. Mitchelmore, *Children's number learning: A research monograph of MERGA/AAMT* (pp. 55-72). Adelaide: Australian Association of Mathematics Teachers.