4 PLACE VALUE ON THE HORIZON

One is hard-pressed to think of universal customs that man has successfully established on earth. There is one, however, of which he can boast—the universal adoption of the Hindu-Arabic numerals to record numbers. In this we perhaps have man's unique worldwide victory of an idea.

—*Howard W. Eves,* Mathematical Circles Squared

Numbers have neither substance, nor meaning, nor qualities. They are nothing but marks, and all that is in them we have put into them by the simple rule of straight succession.

—*Hermann Weyl, quoted in* Armchair Science Reader

ADDITIVE NUMERATION SYSTEMS

How do we comprehend and communicate "how many?"? Although the eye can often perceive five or fewer objects as a whole (see Chapter 3), amounts larger than this need to be counted, or the larger amount needs to be decomposed into smaller amounts that can be subitized and then added. Because we cannot "see" quantities larger than four or five as a whole but instead must operate on them to determine how many, humans across cultures and over time constructed ways to represent amounts symbolically. We invented numerals and operations.

Sticks, Stones, and Bones

The first numerical marks that we are aware of in history come from the Paleolithic era. These marks were slashes, or tallies, carved onto cave walls or into bone, wood, or stone. One slash meant one object; thus ten reindeer were denoted with ten tallies. The marks had a one-to-one correspondence with the objects being counted. Bones bearing such numerical slashes have been found that are nearly 30,000 years old (Guedj 1996).

Because a "one thing, one notch" system was too cumbersome to represent large amounts, over time humans refined this system. One refinement was the use of knots arranged along cords, in Persia, in the fifth century B.C.

(Guedj 1996). By the thirteenth century the Incas had refined this system, developing a *quipu*—a cord held horizontally from which knotted strings hung. The type of knots used, the length of the cord, and the color and position of the strings all communicated levels of quantities: single units, tens, and hundreds. Some cultures used different-shaped stones to represent different amounts, while others made objects out of clay. Sumerian clay stones known as *calculi* (*calculus* is Latin for stone) have been found that date to the fourth millennium B.C. A small clay cone was used to represent a value of one; a ball, ten; and a large cone, sixty. When a contract specifying a particular amount was agreed upon, the calculi representing the sum of that number were placed inside a hollow ball. Notches representing the calculi inside were made on the surface of the seal (Guedj 1996).

Almost all these early number systems used ones, fives, tens, and twenties, but when one realizes that early counting was usually done with fingers and toes, this isn't so surprising.

The Invention of Numerals

The first written mathematical symbols of which we are aware appeared in early Babylonian times (around 3300 B.C.). A nail shape represented units, a chevron shape represented tens. Nine nails and one chevron thus represented nine and ten, the quantity 19. Over time and across cultures, similar written systems were developed. Although new symbols were invented to show quantity groupings, rather than single units, the number of shapes drawn was still in a one-to-one correspondence, either to units or number of groups. And different symbols were used for different-sized groups. For example, the Mayans used a bar to equal five and a dot to represent units. They wrote 19 with three bars and four dots. Ancient Egyptians used lines to represent ones, a basket handle to represent tens, a coiled rope to represent hundreds, and a lotus flower blooming on its stalk to represent thousands. They wrote 19 with one basket handle and nine lines. All of these systems are examples of additive numeration. The operation of addition is employed; the value of the number is equal to the sum of the values of the symbols. Each symbol is repeated the number of times it must be added (Guedj 1996).

Roman numerals are also considered an additive numeration system, in that the symbols represent the worth of the group of objects and the total amount is the sum of the symbols. C represents one hundred; L, fifty; X, ten; V, five; and I, one, and these symbols are repeated the number of times they must be added. One advance exists in this system, though: placing smaller-quantity symbols *before* larger-quantity ones denotes subtraction. That is, XI denotes addition $(10 + 1 = 11)$, but IX denotes subtraction $(10 - 1 = 9)$. This saved a little of the tedium and cumbersome writing when many symbols were needed to represent large amounts. Nevertheless, even with these advances in the writing of numbers (which took thousands of years), the simplest calculation remained arduous.

The Invention of Place Value

In early numeration systems, the value of the digit had little or no relation to the position in which it was placed. Even in the roman numeral system, although placement could denote subtraction, I still meant one, whether it was placed before or after the X. C always equaled one hundred no matter where it was placed: MCI meant one thousand one hundred and one; MCCCI meant one thousand three hundred and one. The amounts designated by the symbols were simply combined.

The positional notation that characterizes our number system today was a big idea in the evolution of number systems. The idea employs the operation of multiplication. For example, the digit 2 in the second column to the left stands for two tens, but when placed in the third column to the left, it stands for two hundreds. No separate symbols are needed to represent tens or hundreds.

The numerals 1 through 9 appeared in India in inscriptions from the third century B.C., but the symbol for none, and the idea of zero, had yet to be invented. The combination of positional notation and the idea of zero in India in the fifth century A.D., which passed via the Arabs to Europe, produced a powerful new system of notation, one that led to advances in calculating and to the development of modern mathematics. In the ninth century the Arab mathematician Muhammad ibn Musa al-Khwarismi wrote a book, *The Book of Addition and Subtraction by Indian Methods*, presenting these new ideas. The book became extremely famous in Europe and was eventually translated into Latin in the twelfth century, thus establishing column arithmetic, using borrowing and carrying, as the method of calculating. Over time column arithmetic became known as *algorism*—the Latin name for al-Khwarismi (Guedj 1996). Today we use the term *algorithm*.

Why Did the Development of Place Value Take So Long?

What makes place value so difficult? Why did it take so long to be developed? For one thing, the idea of zero is conceptually different from all previously developed numbers in that it is not connected to real objects. Piaget noted that the concept of zero introduced number as an idea in itself, separate or abstracted from concrete reality (cited in Guedj 1996). Then, too, the idea of zero evolved in stages. First it was simply functional, a symbol that represented what happened to a number when it was multiplied by ten (324×10 became 3240). Later it was used to stand for the absence of objects in column notation. Only much later in its development did it become a number of its own, defined mathematically as $n - n$ (Guedj 1996).

Children have this same struggle with zero. Recall how Madeline's children figured out that a necklace with 20 beads could be bought with two dimes and no pennies, yet they still thought of 30 and 40 as whole quantities, not as three dimes and no pennies, or four dimes and no pennies. They needed to check out Ellie's conjecture about the zero with other numbers. They did not automatically see that the zero meant no pennies. Children also often write numbers above 100 with two zeroes—using 10013 for one hundred and thirteen, for example. They do not fully understand the combination of the place value columns and the use of zero.

In addition, the idea that a numeral can represent ones or tens or hundreds, depending on where it is placed, involves the big idea of unitizing. The numeral 2 represents two units, but the units themselves can change; they can be ones or tens or hundreds or thousands or more. Cognitively, this is another abstraction from concrete objects. The unit is a variable. Its amount changes depending on the column in which it is placed. The numeral 2 simply represents the cardinality of the units.

HELPING CHILDREN DEVELOP MATHEMATICAL NOTATION

Just as these ideas were difficult for humans to invent, evolving only slowly over many, many years, they are huge developmental milestones in the mathematical development of young children. Martin Hughes (1986) showed children between the ages of three and seven several different cans containing different-sized groups of plastic bricks (one, two, three, five, and six) and asked them to put something on paper to show how many bricks were in each of the cans. The developmental progression he found paralleled the historical progression of the development of numerical writing.

Many of the youngest children made idiosyncratic drawings that seemed to have no connection at all to quantity. They just drew pictures of the objects with no attempt to represent the amount. As the big idea of one-to-one correspondence was constructed, children began to represent quantity with pictographic representations—they actually drew the bricks, one for one, to show the amount. Later the representations took on an iconic representation, with slash marks or dots used as *symbols* to represent the quantity of the objects. Eventually, but not without a great deal of struggle, children attempted to represent the quantity with only one symbol. Understanding that one symbol can represent the whole amount requires an understanding of cardinality. This is a landmark leap for children.

A Window into a Classroom

Jodi Weisbart (2000), a K–1 teacher in New York City, set out in the beginning of a school year to investigate how her children would represent quantity. She began by giving the children various bags of objects and asking them to make signs for their bags to show how many objects they had.

Look at the work of Jack and Susie in Figures 4.1a and 4.1b. Both use a pictorial representation. Jack had a bag with seven teddy bears. He drew every bear in the bag, matching their color: four are green, three are red. Susie had a bag with fifteen Unifix cubes: eight yellow, seven red. Like Jack, she drew every cube, matching color and quantity. On the other hand, some of the children used an iconic representation. For example, Ronald had a bag with fourteen teddy bears, but he drew fourteen circles (see Figure 4.2). Interestingly, he still matches the colors. Most of the children, after drawing pictures or representations of what was in their bag, also added numerals. Were they drawing only because they liked to draw? Or did they need to

FIGURE 4.1a *Jack represents 7 bears pictorially*

represent pictorially what they were beginning to try to represent symbolically? How could Jodi challenge them? What investigation should she plan next?

Historically, it was the need to represent and communicate about larger amounts that led humans to develop numerical symbols. Jodi hypothesized that if she asked her students to represent larger amounts, perhaps they would find the drawing and counting tedious. Maybe they would attempt to represent the amounts with symbols only. She asked the children to make a sign for the door of the classroom so visitors and the principal would know how many children were in the class.

Raquel made a poster with buttons to represent the twenty-eight children—an iconic representation (see Figure 4.3). Because there were so many buttons, however, and now *they* needed to be counted, she wrote counting numerals next to each button. She represented her counting action, but not the cardinality—not the result of the counting. Julie proceeded in a similar fashion but realized that the end result must be represented. "Otherwise visitors would not know which number to look at!" she exclaimed as she wrote *28* at the top of her picture. Eliam decided to draw twenty-eight cubbies to show that there were twenty-eight children (see Figure 4.4). His decision and his drawing show he has constructed an understanding of one-to-one correspondence, but like Raquel he does not represent the result of the counting, only the counting itself. He does, however, count in an organized fashion, top to bottom, left to right. Brycc, on the other hand, used only numerals (see Figure 4.5), writing 28 but also representing the counting. His work is a beautiful example of mathematical development. As children begin to construct a new idea, they often still hang onto the old. As they begin to near a new landmark, we can look back and see evidence of the path they have just traveled.

The large numbers had an even more powerful effect on Ezra, LeeAnne, and Bill. They tired of the counting and attempted to find a way to group the amounts. Here we see the rudimentary beginnings of the use of grouping, rather than counting solely by ones. After several attempts at counting and miscounting, Ezra drew a line across twelve circles and announced, "This is twelve" (see Figure 4.6a). The line was his way of organizing the first twelve into a group so that he didn't have to go back and count them each time.

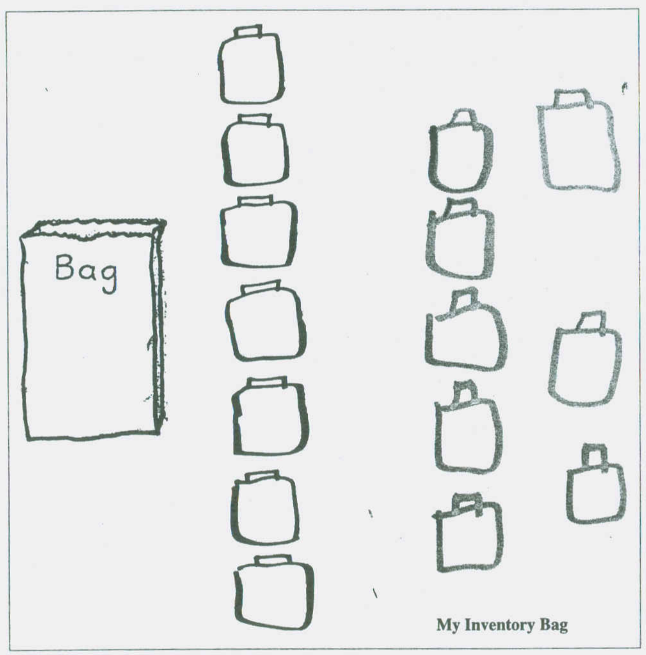


FIGURE 4.1b
Susie represents 15 cubes pictorially

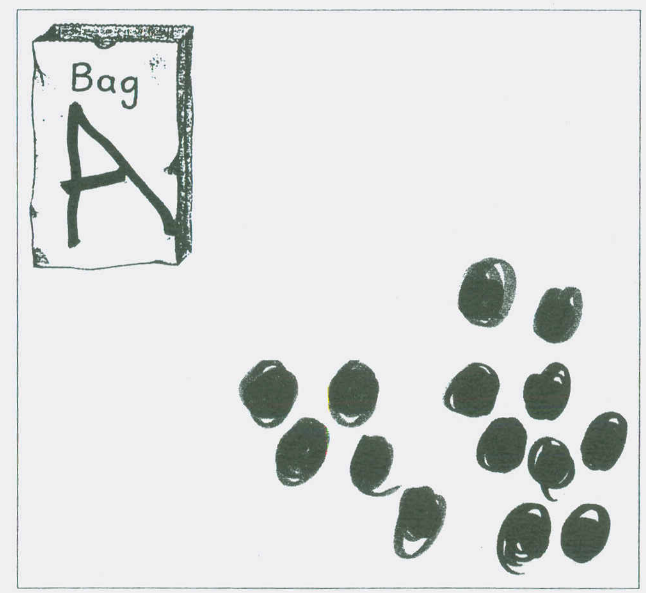


FIGURE 4.2
Ronald represents 14 teddy bears iconically

LeeAnne also organized her work (see Figure 4.6b). Her representation is iconic; she used squares to represent the children. But she grouped them by color into sixes as she worked and wrote, "Sics gren [six green], sics red, sics blos [six blues], sics yelos, for prpols [four purples]." When she completed the drawing she wrote 28. Why LeeAnne chose to use sixes is a mystery. And why did Ezra choose twelve? Often as children begin to try to keep track of their work, they choose numbers that are friendly to them. But as they become more comfortable with our number system, their grouping shifts to fives and tens. Bill's work (Figure 4.6c) represents this shift. He also made an iconic representation, but he adds the numerals *10, 10,* and 8 underneath his cubes. In LeeAnne's and Bill's representations we see the beginning of an additive system. Adding the amounts will produce the total quantity.

Working with Context

One way to use context to challenge children is to manipulate the numbers as Jodi did here, by making them larger. Another way is to build in a potentially realized suggestion or constraint. Of course, just because we design

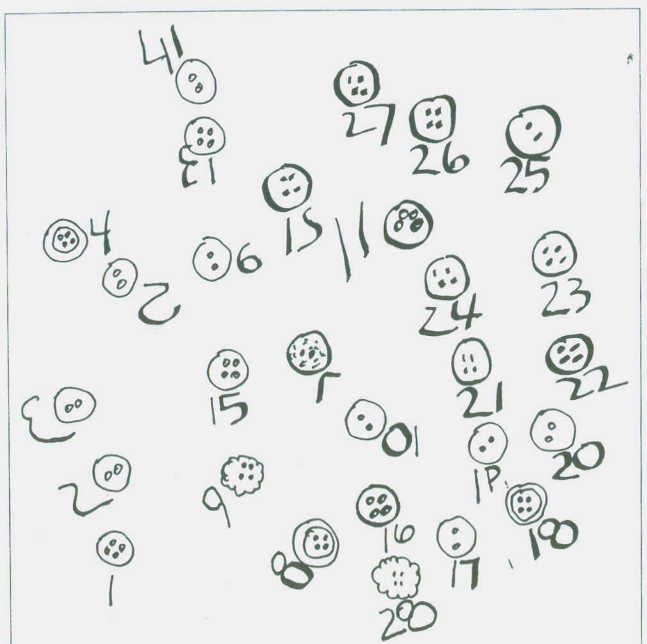


FIGURE 4.3
Raquel represents her counting

a context with suggestions or constraints does not mean our students will experience them as such. Suggestions and constraints themselves must be constructed. Even so, molding the context can be a powerful didactic. To challenge the children to use more grouping, Jodi designed a context with potential grouping suggestions built in. She asked the children to investigate how many cookies would be needed for snack if every child got five and to figure out how many fingers there were in the class of twenty-eight children. For homework she asked them to figure out how many eyes, ears, and fingers there were in their families. In all of these problems children can count by ones if they need to, but the suggestion of grouping (fives, tens, and twos) may lead children to represent the groups.

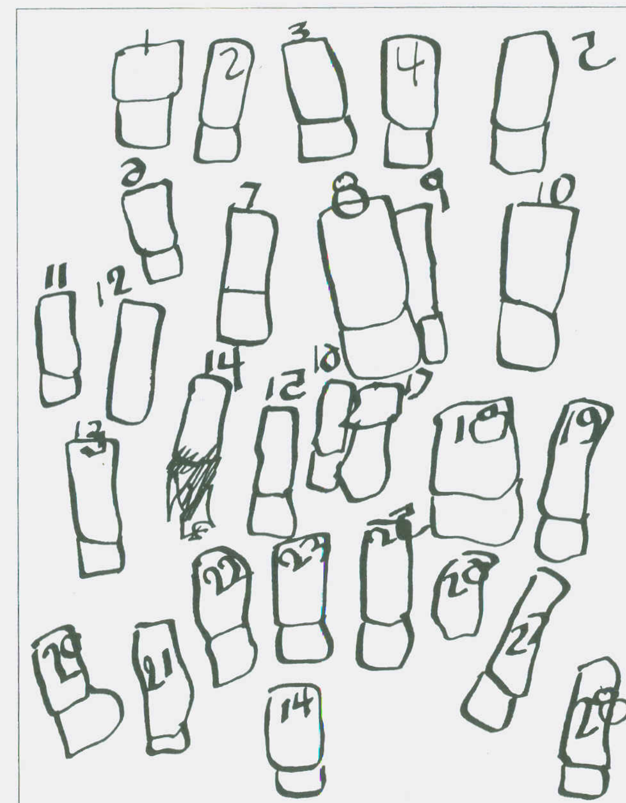


FIGURE 4.4
Eliam represents 28 kids by drawing 28 cubbies

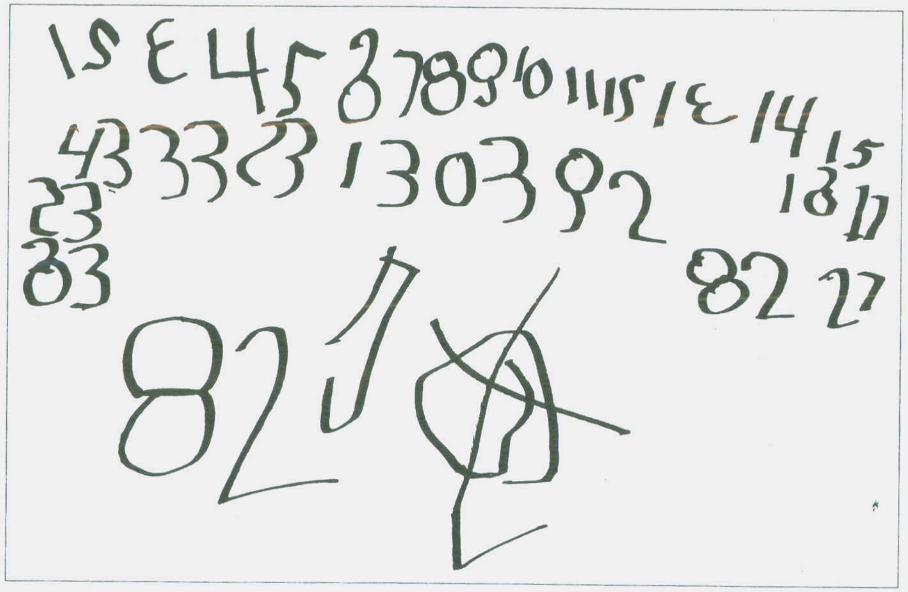


FIGURE 4.5 *Bryce uses numerals only*

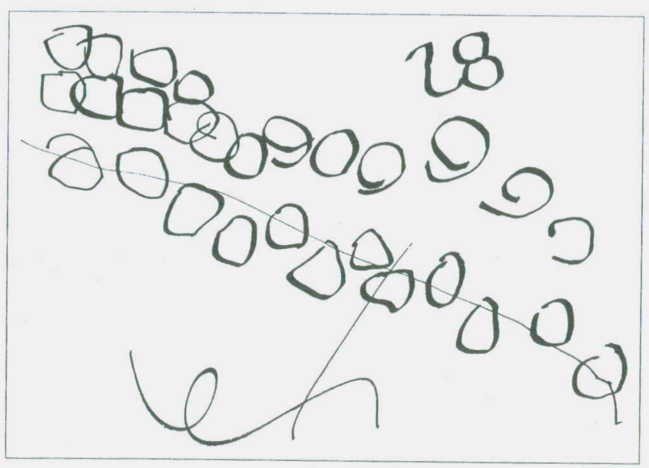


FIGURE 4.6a *Ezra draws a line across 12*

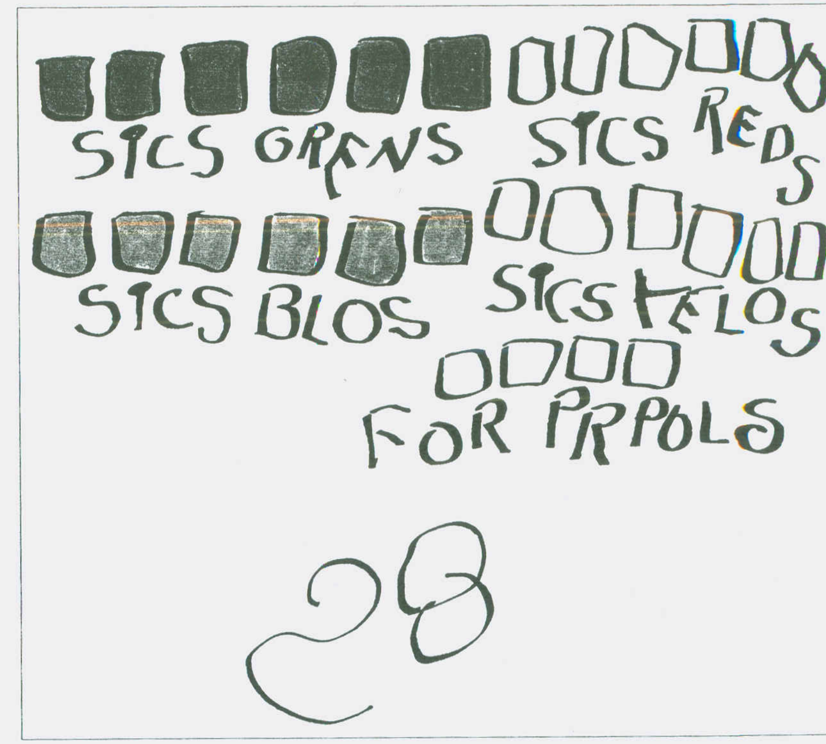


FIGURE 4.6b *LeeAnne uses squares in groups of six*

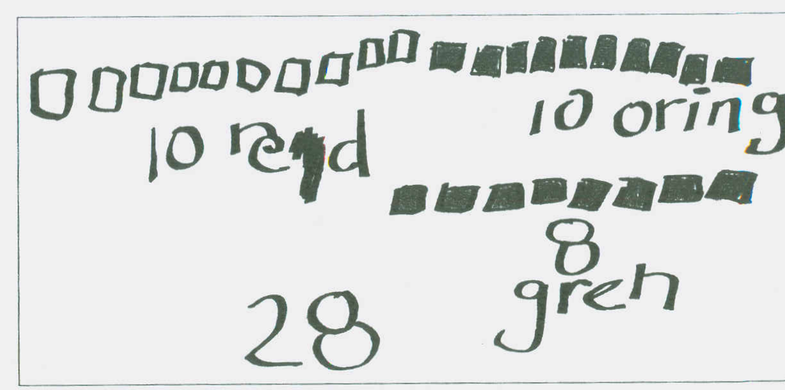


FIGURE 4.6c *Bill groups by ten*

Ivy did just that (see Figure 4.7). She worked on the snack problem by drawing groups of fives. She put two of these groups together and wrote *10*. She continued like this, drawing every cookie, but symbolizing these groups by skip counting by ten until she reached sixty. Then she probably tired from the tedium of all the drawing, because she shifted her strategy and began to represent the five cookies with only the numeral. And she counted these two fives as ten! Ivy has passed a critical landmark: she is now symbolizing, and grouping two fives to make ten.

For the ears and eyes problem, most children drew ears and eyes, although they did draw *the groups* and then label the amounts. The finger problem, however, did result in the shift in representation Jodi was anticipating. For example, Julie used an additive system based on tens: 10 + 10 + 10 = 30. Martin and Everett drew circles and counted on by ten. Underneath the circles they labeled the number of children. They not only used a symbolic strategy but also used numbers in two ways simultaneously—to represent the fingers and the people! One group of ten (one person) equals ten units (ten fingers).

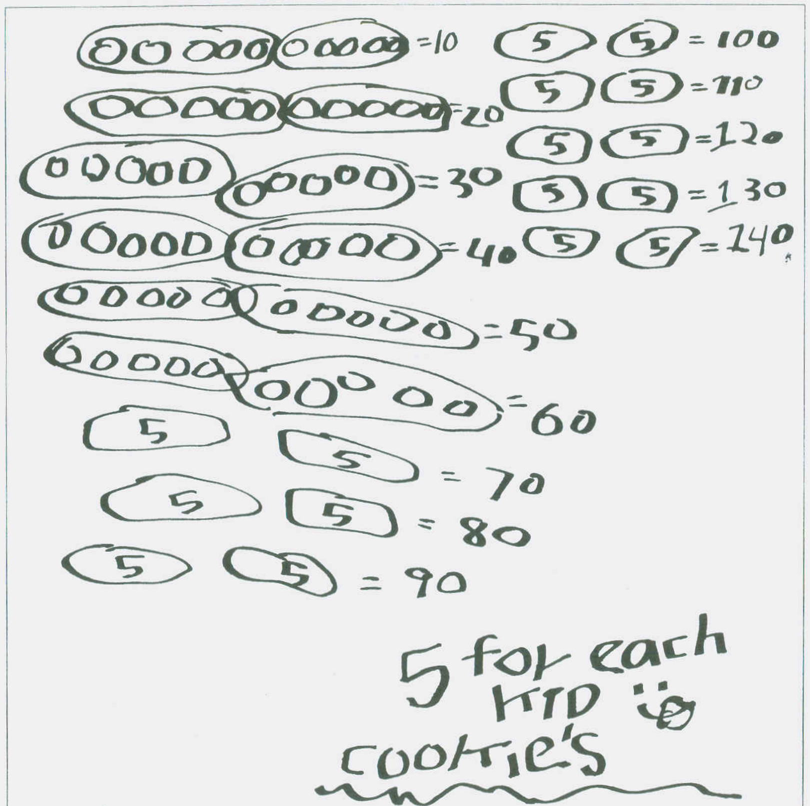


FIGURE 4.7 *Ivy symbolizes and uses two groups of five to make ten*

To support this beginning ability to represent groups of ten symbolically, Jodi next had her students take an inventory of the classroom books. The books used for reading workshop were kept in plastic bins, categorized by author, subject matter, or level. Jodi supplied rubber bands and suggested that the children group the books in each bin into stacks of ten and put a rubber band around each group, and then put a sticky note on each bin indicating how many packs of ten books, how many loose books, and how many books all together (Figure 4.8 shows labels made by Eva and

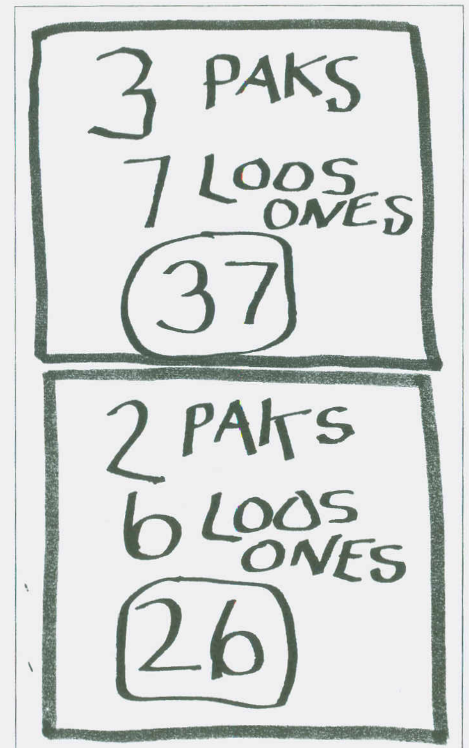


FIGURE 4.8 *Eva and LeeAnne make signs for the bins*

LeeAnne). By the end of this inventory activity, almost all the children were representing numerically and were discussing the place value pattern that was appearing in their representations.

THE DEVELOPMENT OF ORGANIZATION AND A SYSTEM OF TENS

As illustrated in Chapter 3, counting as a way to understand quantity is not a simple activity for young children. Tagging and synchrony when counting, cardinality, and hierarchical inclusion take a long time to develop. Before they begin to construct these ideas, children see no need to rearrange grouped objects with a clear beginning and end, and thus they often recount the same objects many times. As they begin to see the need for organization as a way to keep track, and as they encounter larger groups of objects, they begin to find ways to organize their counting with landmark numbers such as five or ten.

Kamii (1989) reports how children will often say, "Ten, twenty, thirty," etc. louder as they count, in a sense marking ten but not yet making groups of ten. This idea is, of course, insufficient, because if they lose track of where they are as they are counting, it is still difficult to go back and find the group of ten from which to count on. As we saw with Jodi's students, this dilemma is often the impetus for children to find a way to organize objects into groups as they count. And so we begin to see children get to ten, move those objects into a separate pile, then count to twenty, and again move those objects. To check the amount, they skip count by ten: "Ten, twenty, thirty. . . ." Only much later in their development do they think of making groups first, then counting *the groups* by ones and multiplying that count by ten. This latter strategy is based on the big idea of unitizing, an idea Martin and Everett were beginning to construct.

To construct an understanding of unitizing, children almost have to negate their earlier idea of number. They have just learned that one object needs one word—that *one* means one object, that *ten* means ten objects. Now, ten objects are one—one ten. How can something be simultaneously one and ten?

Let's watch children struggling with this idea in a K–1 classroom in Missouri, toward the end of the school year. For several weeks the children have been packaging T-shirts in quantities of ten for a PTA sale and determining how many ten-packs and how many loose T-shirts they would need to fill various orders (for example an order for eighty-three T-shirts would need eight ten-packs and three loose T-shirts). The classroom teacher, Linda Jones, designed the context to support the development of place value.

Today consultant Cathy Fosnot is visiting the classroom, and she and Linda expand the context to include addition. They present the children with several T-shirt orders (for example, one order is for twenty-nine medium T-shirts and seventeen large T-shirts) and ask them to figure out how many T-shirts in total customers should be billed for.

Amy comes up to the chart where Cathy is sitting holding a marker. She faces her classmates and explains, "I left the twenty-nine alone and I had seven. I counted up to twenty-nine. Then I put seven fingers up, and I counted that, and then I put one more on."

Cathy asks, "When you added the seven to twenty-nine, you used your fingers. Can you do that out loud for us?"

Amy holds up seven fingers, but says, "Thirty, thirty-one, thirty-two, thirty-three, thirty-four, thirty-five, thirty-six . . . thirty-seven."

Cathy draws an open number line (see Figure 4.9a) to represent Amy's thinking and asks, "How do we know when to stop?"

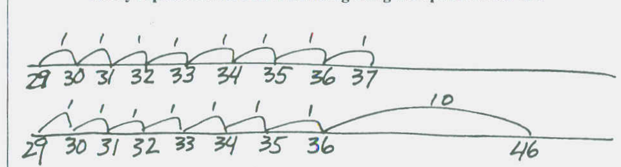
Amy repeats, "I counted up to seven on my fingers."

Cathy draws Amy's attention to the open number line. As Amy uses her fingers once again, Cathy points to each bump of one on the line. They end on thirty-six. "But I have one more to go. I need eight, because there is one more here." She points to the numeral 1 in the 17.

"Is this one?" Cathy asks. "If I do all these seven and one more, will I have done seventeen T-shirts?" Amy tries to explain but gets confused. Cathy attempts to bring the other children into the discussion. "Who can help us? She has done seven so far. How many more does she need to make seventeen?" Several children call out, "Ten"; others begin to count on their fingers from thirty-six.

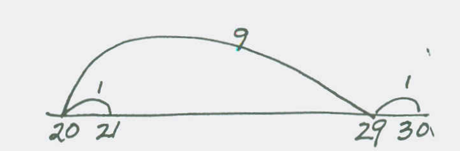
Megan comes to Amy's aid. Counting on her fingers, she says, "Ten more,

Cathy represents children's thinking using the open number line



Amy's and Megan's thinking

FIGURE 4.9a



Stephanie's and Iris's thinking

FIGURE 4.9b

no nine." They check together, as do several other children, until consensus is reached that ten more are needed.

Cathy draws a leap of ten on the open number line, asking, "So how many T-shirts is that now?" Megan and Amy agree that it is forty-six.

Amy begins by adding seven onto the twenty-nine. But rather than calculating the 1 in the 17 as a ten, she treats it as a one and then gets lost. She does not automatically know that the 1 in 17 is one group of ten. She and Megan must count on, using their fingers, in order to figure it out.

Contrast Amy's understanding with that of Stephanie and Iris as we listen to Stephanie explain how she and Iris figured out an order for twenty-one medium T-shirts and nine extra-large ones: "We knew that twenty-one and nine make thirty because we knew that twenty plus nine was twenty-nine and one more is thirty."

"Oh, so you split the twenty-one up into twenty and one, and started with twenty?" Stephanie nods in agreement. Cathy again uses an open number line to represent their thinking (see Figure 4.9b). "And then, like a big frog jump, you went from twenty right to twenty-nine?"

"Yep, and then we put the one back on, and we knew it was thirty," Stephanie concludes with certainty.

Stephanie flexibly decomposes the twenty-one into 20 + 1. Further, she understands that when the nine is added to twenty, the total automatically becomes twenty-nine. She does not need to count on because she understands the number system—how units are added to the tens.

Joey then explains that his order for twenty-five large T-shirts, twenty-five medium T-shirts, and twenty-five small T-shirts can be solved by calculating 20 + 20 + 20 + 10 + 5. Although all the children understand where Joey has gotten the twenties, there is some confusion over whether 20 + 20 + 20 equals 60. Jennie and Sarah help him explain, using coat hangers packed into groups of tens that Linda has provided as a manipulative during these investigations.

Jennie lays out two piles of ten coat hangers and says, "That's twenty: two-packs, like if you take off the zero." She continues taking a pack of hangers at a time and says, "Thirty, forty, fifty, sixty."

Cathy points to two packs of hangers at a time, "So that's twenty, forty, sixty?"

Sarah completes the explanation. "Yes, see you need six packs of tens . . . one, two, three, four, five, six." Then she recounts by tens, "Ten, twenty, thirty, forty, fifty, sixty."

Here the children are beginning to quantify packs. They talk about two packs being twenty and six packs being sixty. Unitizing is in place—two can represent two loose shirts, or two packs of ten shirts for a total of twenty. A multiplicative system rather than an additive one is being used. A landmark on the landscape of learning has been passed.

INVESTIGATIONS TO FACILITATE PLACE VALUE DEVELOPMENT

Place value involves the big idea of unitizing. Any meaningful context that involves children in packing items into groups and keeping track of loose items is likely to bring the idea of unitizing to the surface for discussion. Teachers find various ways to make such packing contexts their own, ways that are connected to the lives of the children in their classrooms.

Several examples have already been described in this book. Madeline Chang has her students build necklaces by alternating five beads of one color with five beads of another color. They charge one cent per bead and make charts for the cashiers that show how many dimes and how many pennies different-sized necklaces will cost. Linda Jones has her students create ten-packs of T-shirts for the PTA sale and asks them to make a chart that will show how many packs and how many loose T-shirts will be needed to fill various orders. Jodi Weisbart asks her students to take inventory of classroom materials.

One day Lisa Merideth, also a teacher in Missouri, saw the school's secretary counting out handouts for each class. But the phone kept ringing, making her lose track. Lisa and her students came to the rescue: they separated the handouts into stacks of ten and made a chart for the secretary showing how many packs and how many loose sheets she needed for each class in the school.

Naomi Cortez in New York City, developed a context around packing Lifesavers and making a chart for the Lifesaver Company, which is in Port Chester, New York. Joy Whitenack and her colleagues (Whitenack et al. in press) developed a context around Aunt Mary's Candy Store.

One year Susannah Blum, a teacher in New York City's East Harlem, helped her students plant a garden outside the classroom window (Blum 1999). Since the children had been busy ordering seeds, Susannah developed a context around a seed company, Seed Time, that specialized in mailing very expensive, hybrid seeds that got sent in envelopes containing an index card with the seeds taped to it in two groups of fives. Susannah hoped that her children might use the quinary structure to think about two fives making a ten and use this to count seed orders by tens or fives rather than ones. The landscape of learning Susannah anticipated over the course of the investigation is shown in Figure 4.10. Note how it parallels the development of early number systems. She anticipated that many of her children would need to draw each seed to be sure they had twenty-eight, but since the context required figuring out how many envelopes the company should send, she also expected they would draw a circle around groups of ten. She further anticipated that strategies would range from little organization, to laying the seeds out in groups of ten at the start, to representing the fives and tens in the envelopes, to representing twenty-eight with two envelopes and eight loose seeds. These are all examples of additive number systems. The horizon in the distance was for children to employ a multiplicative number system characterized by place value—that they would know at the start that twenty-eight was made up of two sets of ten units and eight single units.

Figure 4.11 shows the work of six children early in this investigation as they figure out an order for sixty-three seeds. Jeanne draws envelopes but they do not include any representation of quantity. Dominique draws the envelopes and attempts the structure, but it is pictorial rather than symbolic and he counts by ones as he draws. He finishes with only fifty-two seeds drawn in three envelopes. It is interesting, though, that as he works, his envelope changes from holding eight and eight to holding ten and ten. Perhaps he is beginning to organize his counting into groups of ten as a way to keep track more easily. Jennie begins drawing seeds in groups of ten, loses sight of the need to group, draws and counts the remainder by ones, and ends up with fifty-three. Marcia, Brian, and Danny, on the

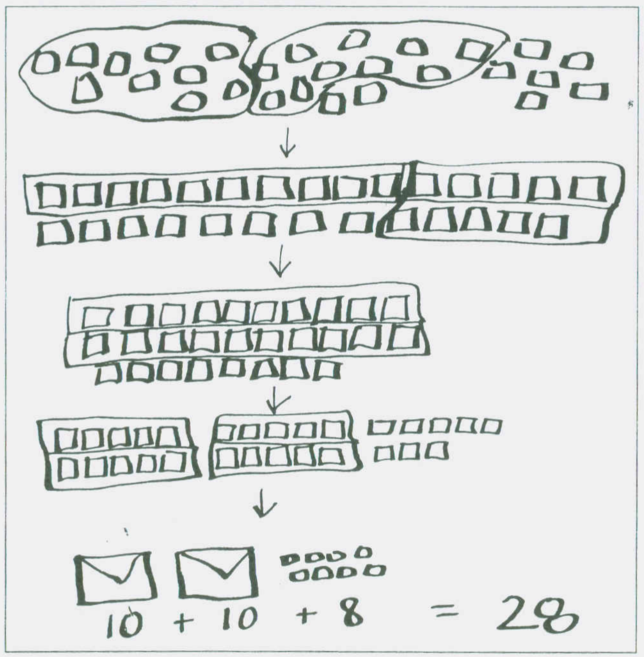


FIGURE 4.10 *Susannah's anticipated landscape of learning*

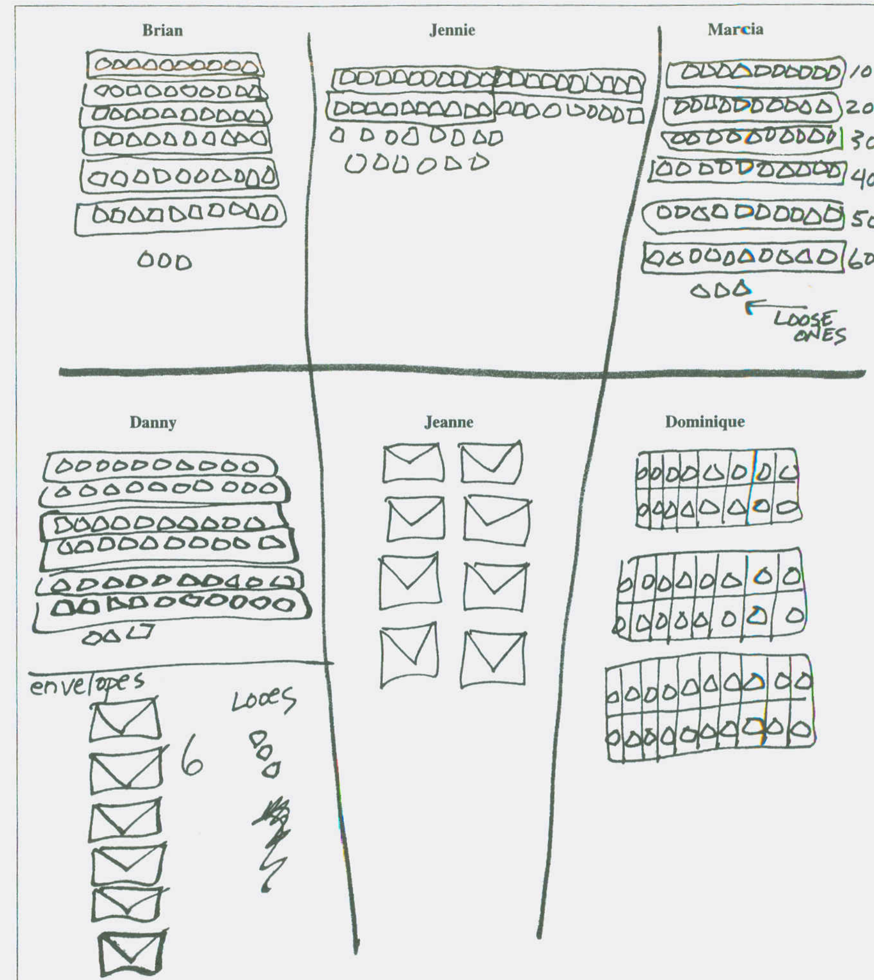


FIGURE 4.11 *Six children's solutions*

other hand, make groups of ten consistently. Marcia keeps track of the quantity as she goes, notating with skip counting (10, 20, 30, and so on). Brian counts the packs when he is done, sure that six packs are sixty. Danny even represents the envelopes and the loose ones using an additive number system.

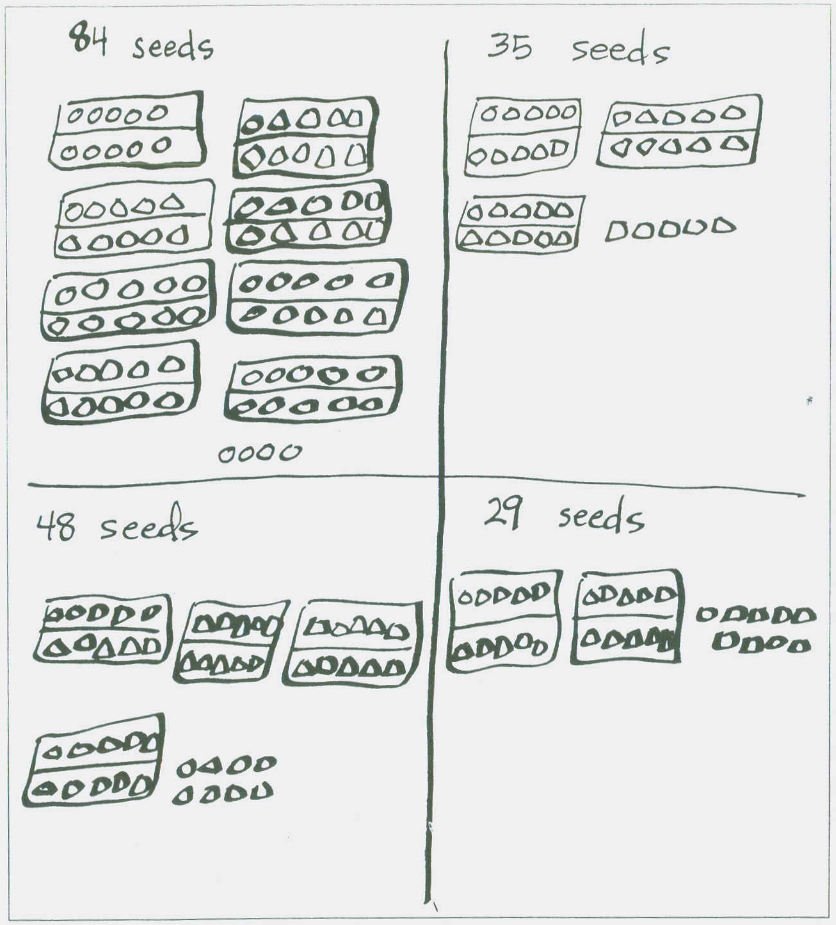


FIGURE 4.12 *Dominique's later work*

As the investigation continued over several weeks, with children packing many different amounts and notating these for the company, strategies began to change. Dominique (see Figure 4.12) is still drawing every seed, but he makes use of the five and ten structure. Jeanne continues drawing envelopes, but she also attempts to keep track of the number of seeds in each (see Figure 4.13). However, this creates a new dilemma for her as she struggles with counting by tens. Note how with 84 she is able to count each envelope by tens; later she becomes confused and reverts to counting by ones as she tries to figure out 48 and 29. Melanie (see Figure 4.14) has clearly understood the ten-seeds-per-envelope structure. And Kenny and Deena

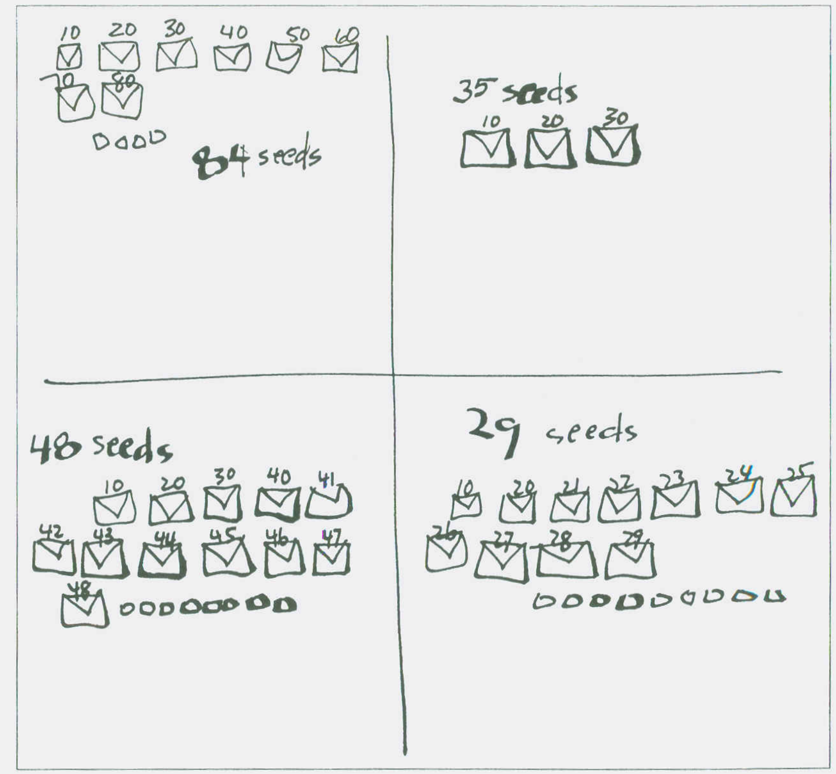


FIGURE 4.13 *Jeanne keeps track of the seeds in the envelopes but struggles with counting by tens*

(see Figures 4.15 and 4.16) construct an additive number system as they work, drawing each seed in the beginning but quickly moving to a system where the envelope represents ten seeds. Note how Kenny has even structured his leftovers into groups of five and how Deena is close to constructing a multiplicative system when she notates with numerals how many of each (envelopes or loose seeds) are needed.

Once children understand unitizing and can anticipate the result of their grouping activities, the horizon shifts and new landmarks appear. They need to explore how many more loose items are needed to get to the next full pack. If children have made charts that show the number of packs and the leftovers, they can add to this chart a third column for the number of loose items needed to make another pack (see Figure 4.17). Note that the numbers in the right two columns add to ten. This idea, that ten must be the result of adding on more loose items to the already loose items if one wants only full packs, will help children when they begin computation with double digits. For example, when adding 38 + 13, an important strategy (see Chapter 8) is to keep the 38 whole, add 2 to get to the landmark number of 40, and then add on 11 more, in leaps of 10 and 1. An alternate strategy is to add on the tens first and then units—for example, 47 + 24 = 47 + 20 + 3 + 1.

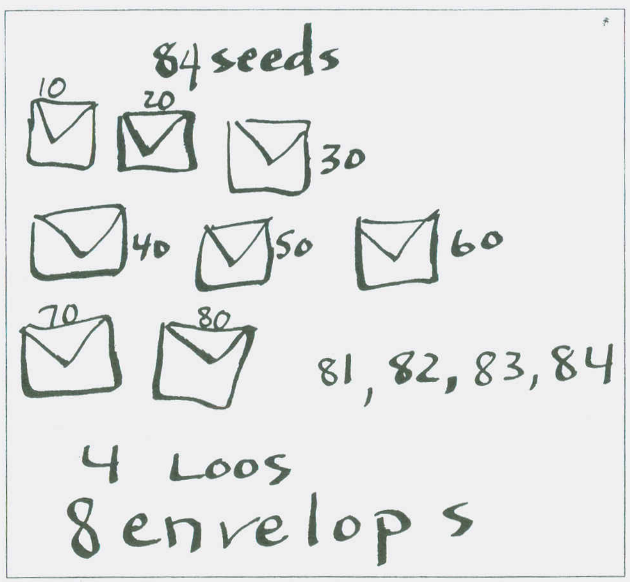


FIGURE 4.14
Melanie counts the seeds in the envelopes by tens and then the leftovers by ones

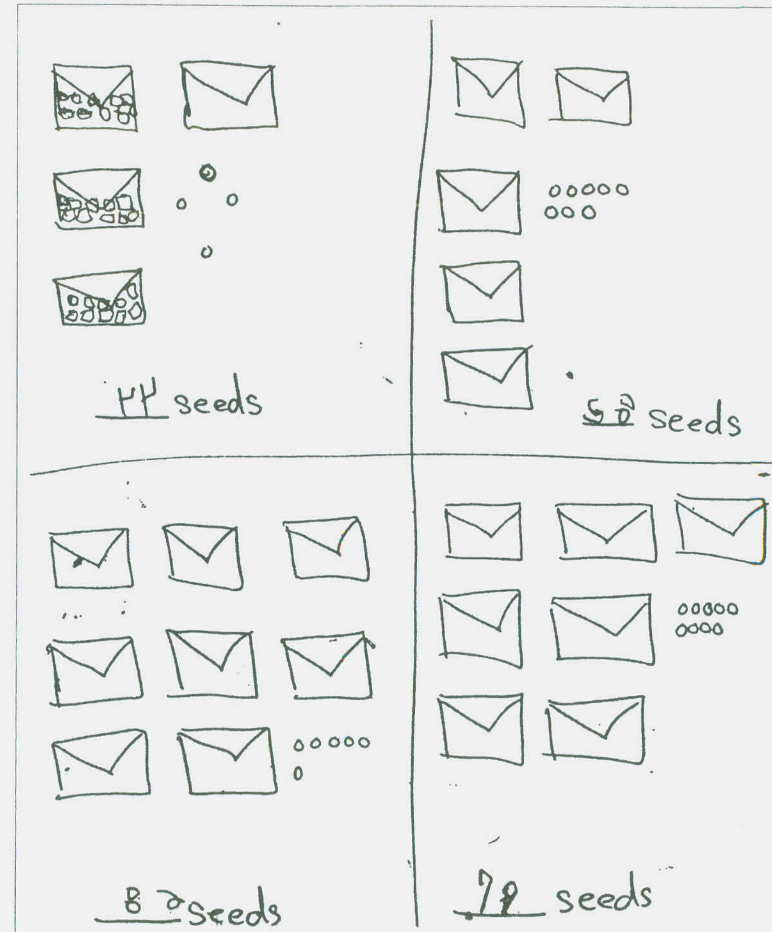
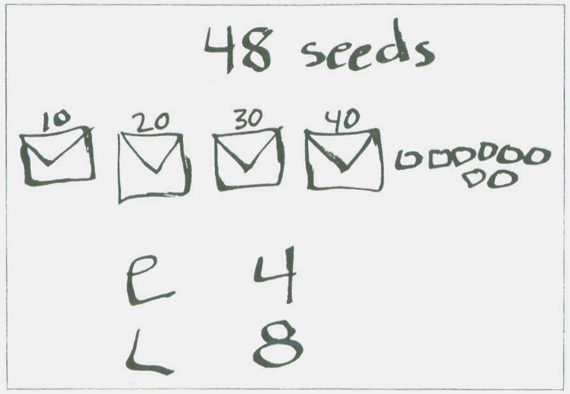


FIGURE 4.15 *Kenny constructs an additive number system*

FIGURE 4.16
Deena constructs an additive number system



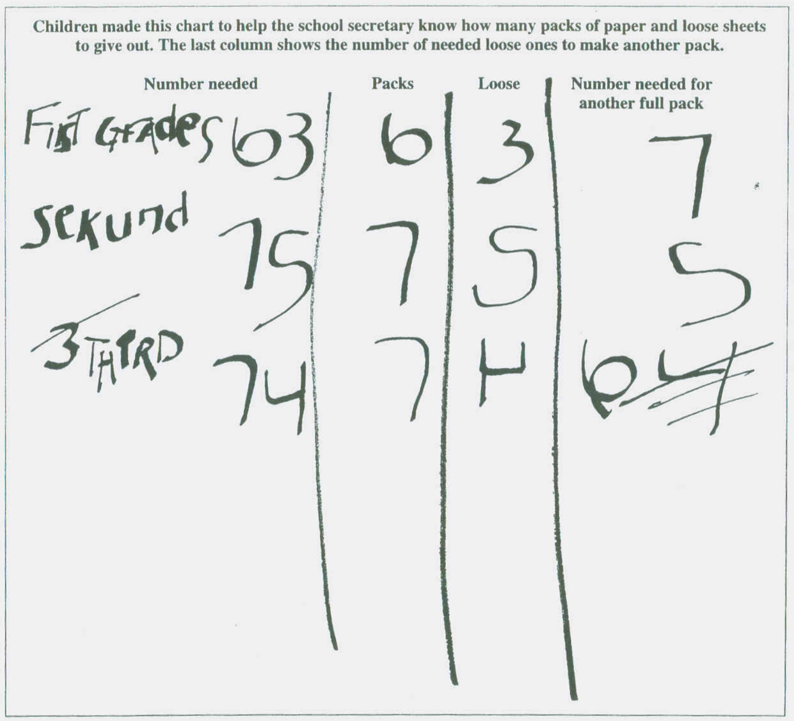


FIGURE 4.17 *Loose items, packs chart*

SUMMING UP . . .

Developing an understanding of our number system parallels the historical development of number systems. As Hermann Weyl says in the quotation used as an epigraph for this chapter, "Numbers have neither substance, nor meaning, nor qualities. They are nothing but marks, and all that is in them we have put into them by the simple rule of straight succession." Just as human beings put quantitative meaning into symbols over time, children's symbol usage moves from tallies to groups to numerals—from additive systems to multiplicative systems.

Unitizing is the big idea at the heart of place value. Critical addition and subtraction strategies in relation to place value are adding or removing units to get to the next ten (adding 3 to 27 to get to 30 or removing 7 from 27 to get to 20) and adding or subtracting in leaps of ten (16, 26, 36 or the reverse). All of these ideas can be explored in packing contexts like the ones described in this chapter.

As children grapple with these ideas, they construct important mathematical ideas that took centuries to develop. Consider the other epigraph to this chapter: "One is hard-pressed to think of universal customs that man has successfully established on earth. There is one, however, of which he can boast—the universal adoption of the Hindu-Arabic numerals to record numbers. In this we perhaps have man's unique worldwide victory of an idea." When it took humans so long to construct these ideas, how can we not be impressed with the seriousness of young children's mathematical endeavors, their struggles to invent such big ideas, and their capacity to mathematize?