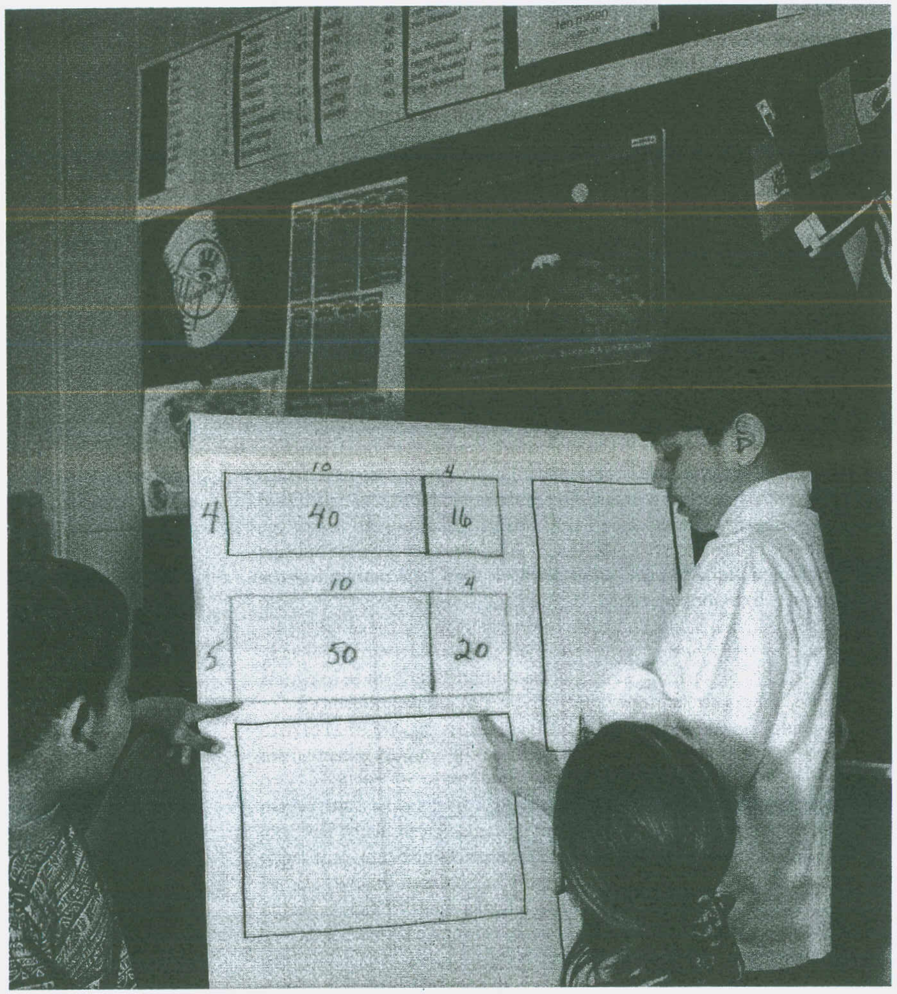


7 DEVELOPING EFFICIENT COMPUTATION WITH MINILESSONS

> *I think, therefore I am. . . . Each problem that I solved became a rule which served afterwards to solve other problems.*
>
> —*René Descartes*

> *Mathematics is the only instructional material that can be presented in an entirely undogmatic way.*
>
> —*Max Dehn*

MINILESSONS WITH MENTAL MATH STRINGS

"I knew that six times six was thirty-six, so I took away six. That gave me thirty." Diana, a New York City fourth grader, is explaining how she solved the problem 5 × 6. Her friend Linda, who is sitting next to her, agrees with this answer but shares a different strategy. "I knew ten times six was sixty, so half of that is thirty." Most of the other children know the fact by heart, and Grady Carson, their teacher, continues with his string of problems. He writes 30 × 6 next. Randy explains that he added 30 and 30 to get 60. Then he added 60 three times to get 180.

Grady is beginning math workshop, as he normally does each day, with a short ten- or fifteen-minute minilesson focusing on computation strategies. In contrast to investigations, which characterize the heart of math workshop, the minilessons are more guided and more explicit. They are designed specifically to highlight certain strategies and to develop efficient mental math computation. Each day, Grady chooses a string of four or five related problems and asks his students to solve them. Together they discuss and compare strategy efficiency and explore relationships between problems.

Crucial to Grady's choice of problems is the relationship between them. He picks problems that are likely to develop certain strategies or big ideas that he knows are important because they are landmarks on the landscape of learning. We call these groups of problems *strings* because they are a structured series of computation problems that are related in such a way as to develop and highlight number relationships and operations. Designing such strings and other minilessons to develop computation strategies requires that teachers have a repertoire of strategies for multiplication and division and that they know how to play with numbers.

Choosing the Strategies, Choosing the Numbers

The string that Grady is using is shown in Figure 7.1. Grady has chosen these numbers to encourage children to use the distributive property, which is the basis for why the traditional multiplication algorithm works. When we multiply 12×13 using the algorithm, we are actually multiplying first by 2 and then by ten—$2 \times (3 + 10) + 10 \times (3 + 10)$. The twelve groups can be broken up into different parts, each part multiplied by 13. As long as all twelve groups of thirteen are accounted for, the sum of the answers to the parts will be the answer to the whole: $(6 \times 13) + (6 \times 13) = 12 \times 13$; $(4 \times 13) + (8 \times 13) = 12 \times 13$. When children are just taught the procedures of the algorithm, which require them to treat each number as a digit regardless of its quantity, they lose sight of the arrays they are dealing with and an understanding of the distributive property is often sacrificed. (Remember how Sophie [in Chapter 6] could not multiply 11×11 with the algorithm but knew 10×11 and 1×11?)

Grady wants to develop children's ability to use the distributive property with understanding. He does this by using a string of problems that will bring the distributive property to the surface for discussion. Note the relationships in Grady's string. The answers to the first two problems when added produce the third. The next three problems are similarly related.

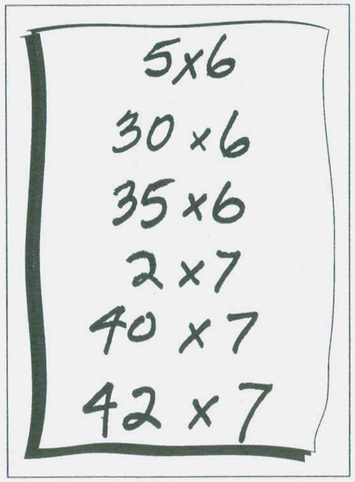


FIGURE 7.1
Grady's String

Although Grady has thought about the problems beforehand and has a string of related problems ready, he does not put all the problems on the board at once. Instead he writes one at a time, and children discuss their strategies before the subsequent problem is presented. This way, the children can consider the strategies from the prior problem as well as the numbers, and they are prompted to think about the relationships of the problems in the string as they go along. Sometimes, depending on the strategies he hears, Grady adjusts the problems in his planned string on the spot to ensure that the strategies he is attempting to develop are discussed and tried out. Let's witness this in action.

Grady continues with 35×6. "Becky?"

"I added thirty-five six times," Becky explains. "But to make it easier I added thirty-five and thirty-five to get seventy, like Randy did before. Then I doubled seventy and added another. That gave me 210."

Steve nods his head in agreement but continues with a smile, "I have a shorter way. The last two problems add up to the answer!"

Grady asks him to explain.

"I split thirty-five into thirty and five, and I did thirty times six first. Then I did five times six, and I added them together."

Grady draws the open array shown in Figure 7.2 to represent Steve's steps and help the other children visualize the parts. "That's pretty neat, isn't it? Who can explain his strategy? Diana?"

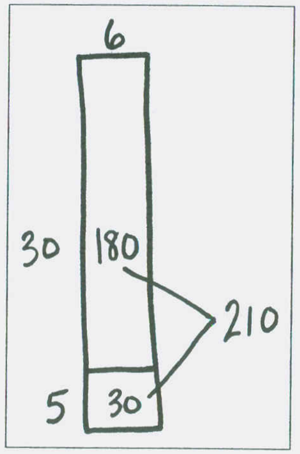


FIGURE 7.2
Steve's Strategy for 35 × 6

"He broke thirty-five up into two parts, thirty and five, then he multiplied each piece," Diana says, quite succinctly.

Surprised that the pattern has been noticed so quickly, Grady skips the next two problems in his string and goes immediately to 42 × 7. He wants to see whether the children can find the component parts themselves. "Can we use Steve's strategy for this one?"

Robert offers an answer of 280, and Grady asks him to explain what he did.

"I knew four times seven was twenty-eight, so forty times seven is 280." Robert has used the associative property for this part of his solution, but now he looks puzzled. "Uh. . . ."

Grady draws the open array for 40 × 7 and labels it 280. "What did you leave out?"

"Oh. yeah, two times seven."

Grady completes the open array (see Figure 7.3). "This is such a neat strategy that you came up with, Steve. How about if we make a sign to post with our other strategies. Steve, how can we phrase it?"

"Split the two-digit number into tens and ones," Steve offers as the first step. "And then multiply each number by the other number."

"What do we think of that?" Grady throws the description back to the class to ponder.

"How about if we say 'factor' for 'the other number'?" suggests Diana.

"Oh, nice," Grady says. "I love it when you use specific mathematical language."

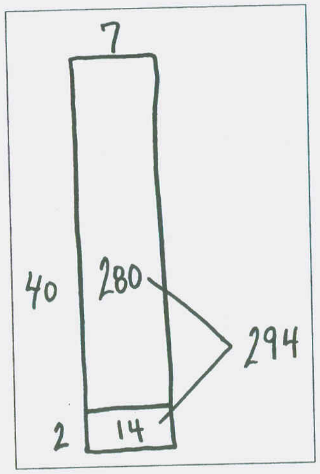


FIGURE 7.3
Robert's Strategy for 42 × 7

Although Grady had not intended to focus on writing a description of the strategy, mathematics is also involved in this process. As children grapple with language that is specific, that clearly communicates what they want to say, they come to appreciate the need for mathematical language. They grapple with definitions and mathematical terms.

"Let's use the word *factor* again," Grady suggests. "How about, 'split one factor into tens and units and multiply each part by the other factor'? And then what is the answer called?"

Several children chorus, "The product," and Grady adds, "That gives you the product." The sign now reads, "Split one factor into tens and ones and multiply each part by the other factor. That gives you the product." Grady posts it with other signs listing strategies discussed in earlier minilessons.

Just because one child has constructed a strategy does not mean all the other children in the class understand it and can use it meaningfully. Posted signs have the danger of becoming a list of algorithms that children will adopt as rote procedures. Grady wants to ensure that his students understand the strategies dealt with in his minilessons. Therefore, he concludes this minilesson with three more problems (25 × 9; 26 × 9; 46 × 5) and suggests that the children try them using Steve's strategy, recording the problems in their journals, drawing arrays, and then sharing their work with a partner.

Tools, Representations, and Models

By asking the children to draw arrays as they try out Steve's strategy, Grady has them model the problem. He used this same process when he drew an array for 40 × 7 for Robert. The piece missing, 2 × 7, became apparent. Of course, for this modeling to help, children need to understand that the array comprises rows and columns. Tools such as square tiles and graph paper, employed while investigating contexts like those described in previous chapters, are essential to help children develop an understanding of arrays. Grady's students have already investigated many arraylike contexts.

Here, as Grady's students represent their strategies on paper, they are able to check out whether they have done all the parts. They are using the array as a model to think with—as a tool. Understanding the distributive property is a big idea. Often children confuse the factors they have done and the factors they still need to do. The part/whole relationships can be difficult to keep track of mentally. Two of Grady's students have just this struggle as they discuss solving to 26 × 9 based on their solution to 25 × 9.

"You just add one more."

"But one more what? A twenty-five or a nine?"

"Twenty-five. The answer's 250."

"No, I don't think so. Twenty-five times nine is twenty-five nines. We need twenty-six nines. So we need one more nine." They finally succeed in drawing the arrays shown in Figure 7.4. "See, when you added twenty-five, that was twenty-five times ten."

"Oh, yeah. Boy, this is confusing, isn't it?"

While doing strings, Grady (like Miki in Chapter 5) has used the open array as a model to *represent* children's strategies. This visible representation focuses discussion. Over time, children move from using the graph paper array as a tool, to using an open array to represent their thinking, to mentally using an array as a model for thinking.

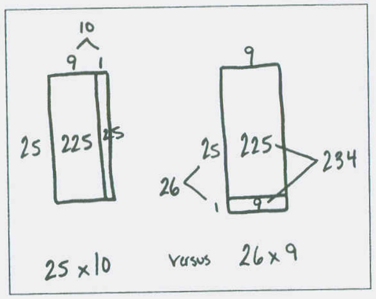


FIGURE 7.4
Children's work
25 × 10 versus 26 × 9

DEVELOPING MULTIPLICATION AND DIVISION STRATEGIES

Extending the Distributive Property

The distributive property can also be used in connection with subtraction. For example, 49 × 7 can be solved as (50 × 7) − 7. Peter Markowitz's fifth graders named this strategy "friends, more or less"—meaning they could use a friendly number, like 50 or 100, and then add or subtract as needed. Let's watch this strategy in action.

Peter writes 48 × 7 on the board. "Treshaun?"

Treshaun makes use of fifty. "I knew one hundred times seven was 700, so fifty times seven is 350. Then I subtracted fourteen."

"And for ninety-eight times thirty-two??

"That's one hundred times thirty-two minus sixty-four. Friends, more or less." (See the array in Figure 7.5.)

The distributive property can also be used when the numbers can't be made friendly so easily. Peter's fifth graders call this strategy "the ugly one," because it has so many parts. For example, 37 × 84 might be solved as (30 × 80) + (7 × 80) + (30 × 4) + (7 × 4)—see the array in Figure 7.6.

Using the Associative Property

Understanding that the order in which the factors are multiplied doesn't affect the result is a big idea for children and must be constructed through investigations such as the Christmas ornaments investigation Hollee Freeman uses in Chapter 3. Even after children understand this idea, though, they do not automatically use it when computing. Mental math strings can be used to help children develop computational strategies based on this idea. Let's return to Grady Carson's class on another day.

Grady begins his minilesson with the problem 10 × 6 = ?. All the students are clustered around him, math journals in hand. Grady assumes no one will have trouble solving this problem, and he will use it as the basis for the strategy he is trying to develop. He calls on Caroline and is not surprised by her answer.

"Sixty," Caroline responds quickly. "I just know that one."

Other children nod in agreement, and Grady continues with his string, writing 3 × 6 underneath 10 × 6 = 60. Once again, this is easy. Several children respond, "Eighteen."

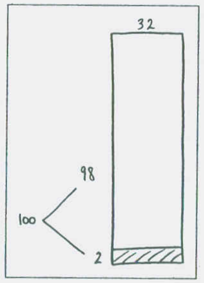


FIGURE 7.5
Friends More or Less
98 × 32 = (100 × 32) − (2 × 32)

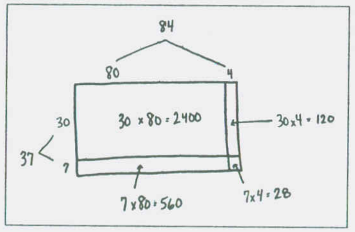


FIGURE 7.6
Array for 37 × 84
The Ugly One

Grady adds 18 as the product and adds a third problem to the board, 30 × 6. The children can't solve this problem as readily. They take more time to figure it out. Grady waits a bit, then asks for solutions and strategies. "Randy?"

"One hundred and eighty."

"How did you do it?" Grady inquires.

"I added ten times six, which was sixty, three times."

Grady draws the open array in Figure 7.7 and writes 3 × (6 × 10) underneath.

John comments, "It's also three times six is eighteen, so just add a zero for thirty times six."

"Did you add a zero to eighteen? Is three plus zero thirty?"

John looks puzzled but shakes his head. "No. I just put the zero on the end. I didn't really add it."

"So that's a pattern you've noticed," Grady acknowledges and writes (3 × 6) × 10.

"Let's try another." Grady writes 4 × 7.

"Twenty-four . . . no, twenty-eight," responds Robert. "Two times seven is fourteen, so I doubled it."

Grady adds the answer and writes 10 × 7. Again, this is easy, and Grady writes in the 70. He follows with 40 × 7 to ensure that the pattern resulting from the associative property—4 × (7 × 10) = (4 × 7) × 10—will appear.

Several children immediately use the pattern. "Two hundred and eighty. It's four times seven with a zero on the end."

"We've been using a pattern that John and several others noticed as we worked through this string," Grady summarizes. "But, of course, to be sure it will always work, we have to figure out why it is happening. For homework, draw some arrays. See if you can figure out why it is happening. Why is it, if you are multiplying, say, sixty times eight, you can do it by multiplying six times eight and putting a zero on the end?"

In this minilesson Grady is highlighting the associative property. He has done this by choosing problems whose answers are similarly patterned. Although his fourth graders can now solve problems like 40 × 7 easily, they may or may not fully understand why it works to multiply (4 × 7) × 10. However, as they investigate this strategy in their journals, Grady will be able to analyze their thinking about this topic and respond to it. The information he gleans will help him determine what to do next as he helps his students move through the landscape of learning.

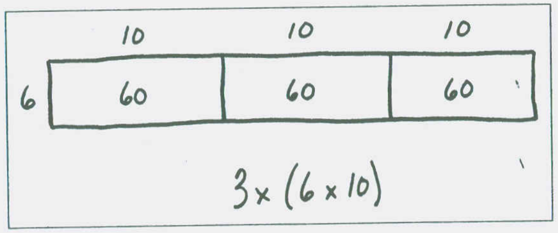


FIGURE 7.7
Randy's Strategy

Doubling and Halving

This strategy is directly connected to the associative property, but because it is so important, we are highlighting it separately. As we describe in Chapter 3, children begin to use doubling early on. To figure out 4 × 7, for example, they do 2 × 7 and double it. Making use of landmark numbers and then doubling or halving is a powerful strategy for double-digit computation as well. To calculate 4 × 35, we might double 70; to calculate 50 × 42, we might take half of 4,200 (see Figure 7.8). Developing these strategies using problems strings and open arrays is easy. As long as the problems make use of this pattern, the arrays simply get cut in half or doubled. A sample string for developing these strategies is 2 × 24, 4 × 24, 8 × 24, 8 × 12, 4 × 12, and so on.

The strategy becomes even more powerful when doubling and halving are used simultaneously. For example, 3½ × 14 can easily be solved by doubling the 3½ and halving the 14: 7 × 7. The array in Figure 7.9 shows why this strategy works. It is based on the associative property, because the doubling and halving is a result of how one associates the factors—3½ × (2 × 7) = (3½ × 2) × 7. Once children understand what happens to the product when one factor is doubled *or* halved, the string can be expanded to include problems where doubling *and* halving are used simultaneously. A slight variation on the string above brings this relationship to the surface: 2 × 24, 4 × 24, 2 × 48, 8 × 12, 16 × 6, 32 × 3, and so on.

This strategy can be generalized to any number and its reciprocal: 3, ⅓; 10, ⅒; and so on. A messy problem like 18 × 3⅓ can be solved by taking a third of eighteen, and multiplying three and a third by three. This results in 6 × 10, an easy problem to solve. Similarly, .8 × 30 can be solved by 8 × 3. Strings can bring these patterns to the surface, and children can then investigate why the strategy works by drawing arrays on graph paper.

Using Money

Because money is such a powerful and ever present context in children's lives, it can be used to develop landmark numbers like 25, 50, and 75. Let's listen as Carol Mosesson's third graders discuss their multiplication strategies based on the use of money. Zenique is sharing how he came up with 190 as the answer to 20 × 9.

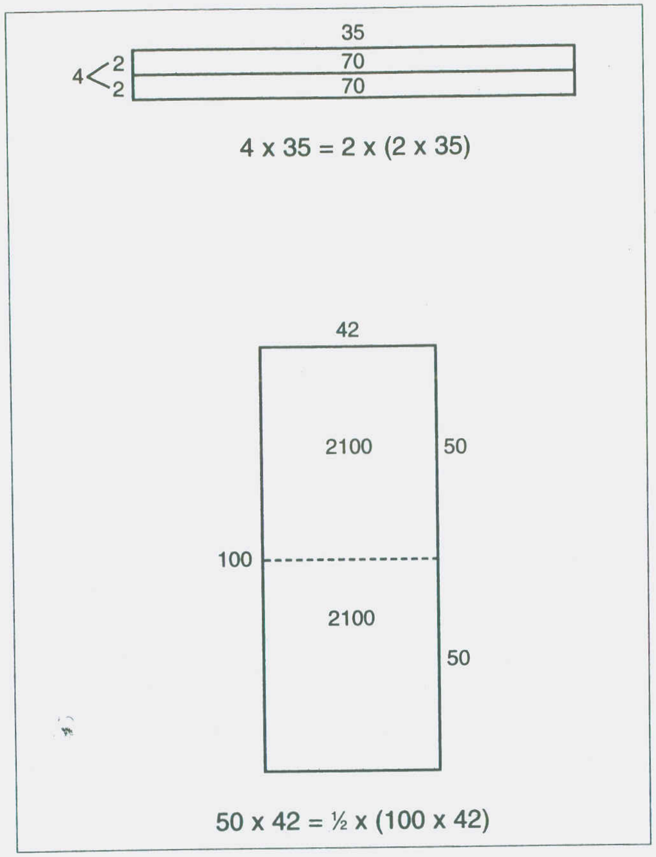


FIGURE 7.8

"Five twenty-dollar bills is a hundred dollars," he begins, "so another five is two hundred. Oh, I know what I did wrong. I only took away ten. I should have taken away a twenty-dollar bill. It is 180," he concludes, agreeing with an answer given by another classmate.

"Money is a clever way to think about that problem, Zenique." Carol writes 25 × 9 on the board. "How about this one? Shakira?"

"Three hundred . . . no, 225. I counted by twenty-five."

"How many twenty-fives would it take to make three hundred?" Carol probes.

Shakira answers quickly, "Twelve."

Carol is surprised. "Wow, how did you know that so fast?"

"I knew four quarters made a dollar, so times three, that's twelve."

"That's exactly how I did it," Olana joins in. "I got 225 because I knew that four times twenty-five was a hundred, because that's like four quarters. So another four times twenty-five is another dollar. And one more quarter is 225.

To develop this type of thinking, teachers can begin a minilesson using real coins, or pictures of coins, in an array. For example, if a 4 × 4 array of quarters is shown, many children will explain that they know that each row is a dollar. Because quarters are worth twenty-five cents, the problem can then be written as 16 × 25 and strategies developed—16 × 25 = 4 × (4 × 25), for example. After several minilessons based on money, children are able to use it as a context, even when the numbers are bare, like in strings.

Using Fractions

Once children develop a sense of landmark fractions, like ½, ¼, and ¾, using them can also become a powerful strategy. For example, 75 × 80 can easily be solved by thinking of the problem as ¾ × 80. You only have to remember to compensate for the decimal in the answer (multiply 60 by 100, because you treated 75 as 75/100).

Minilessons can be designed to develop this ability. If the problems in the strings progress from fractions to decimals to whole numbers, children quickly see the resulting patterns. For example, a string like ¼ × 80, .25 × 80, 25 × 80, ½ × 60, .5 × 60, .50 × 60, 50 × 60 produces the appropriate patterns in the answers, and children can use arrays to explore the relationships.

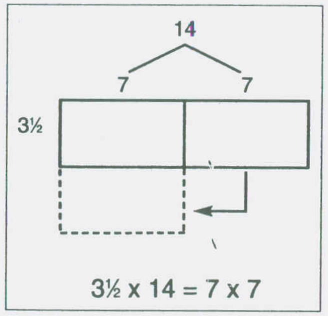


FIGURE 7.9

USING THE OPEN ARRAY WITH DIVISION

Division is defined, mathematically, as the inverse of multiplication. Traditionally, when teaching the long division algorithm, teachers have encouraged children to think about division as a "goes into" action. For example, seventy-five divided by five was taught as five goes into seven once, with two left over, which is joined with the five, and five then goes into twenty-five five times, for an answer of fifteen. Teaching division as a "goes into" action is an insufficient model, and treating digits separately is confusing and often makes little sense to children.

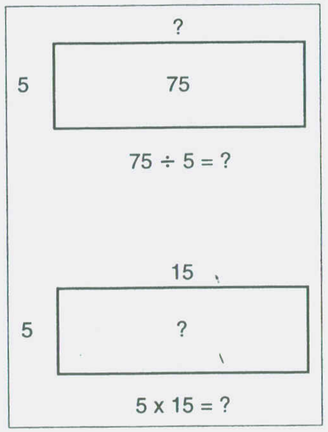
When children are allowed to construct their own division strategies, they often use multiplication, building up to the whole rather than subtracting from the whole. The open array represents both repeated addition and subtraction strategies well, and it develops the connection between multiplication and division. The pictures are the same for multiplication and division, but the knowns and unknowns are different (see Figure 7.10).

Reducing

Let's watch Andrea Franks use an open array with a division string to develop a reducing strategy with her fourth and fifth graders in New York City. Andrea has written 24/6 on the chalkboard.

"Four," Abbie, a fourth grader, states with confidence. "I just knew that one."

FIGURE 7.10



All Andrea's students know this fact by heart, so she moves on, writing 48/12 underneath it. "Sue?"

Again, this is an easy problem. "Four. I knew that twelve times two is twenty-four, so I doubled: twenty-four times two is forty-eight."

Andrea draws the open array shown in Figure 7.11 and asks other students to paraphrase what Sue did. Then, continuing her string, she writes 48/6. Children's answers vary. Some say two; others, eight. Interestingly, although most of the children know the multiplication fact 8 × 6, they are so intent on looking for relationships in the string they don't think about it right away. Andrea asks Anton, who has answered two, to explain.

"You cut the twelve from the last problem in half, so I halved the answer."

As is common as children try to apply the doubling and halving they have used in multiplication to division, Anton has halved the answer when he should have doubled it. Andrea draws an array of his thinking: the array remaining after halving twelve to six and halving four to two produces an area of only twelve, not forty-eight (see Figure 7.12).

Afrique, a fifth grader who has gotten eight as an answer, explains, "No, Anton, you need to move the right part of the array down, underneath the four-by-six array." Andrea draws the rearranged array (see Figure 7.13).

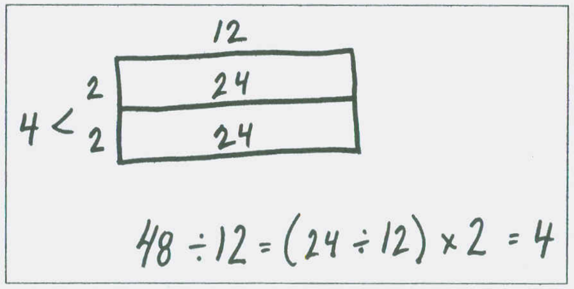


FIGURE 7.11
Sue's Strategy

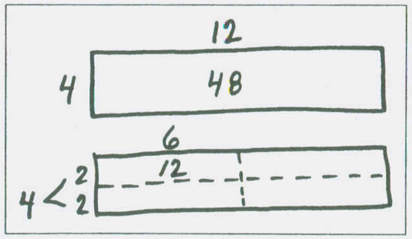


FIGURE 7.12
Anton's Strategy

Ray, who has gotten two as his answer, speaks up. "I don't get why you're moving that down, Afrique."

"Because you need it to be six, not twelve, but you still need the inside to be forty-eight." Afrique's reasoning convinces Ray, as well as the rest of the children.

Andrea goes on with her string, this time writing 96/12. After giving the students time to think out the relationship, she calls on Verona, one of her fourth graders.

"Eight," responds Verona. "Because forty-eight divided by twelve is four. Ninety-six divided by twelve is double, so it has to be eight." Andrea draws the open array shown in Figure 7.14.

Afrique volunteers a different way. "I used forty-eight over six," she explains, as Andrea represents her thinking by drawing the array shown in Figure 7.15. "But I doubled both, so the answer is the same. Eight."

"But didn't you double both, too, Verona?" asks Ray.

"No, because I didn't start with forty-eight over six. I started with forty-eight over twelve. Only the forty-eight doubled."

Andrea is aware that Ray and probably other students are still confused. She asks them to consider each of the arrays and to compare Verona's strategy with Afrique's. "How many columns? How many rows?"

Lena responds, "There's twelve columns and eight rows in both. And they both doubled the forty-eight."

FIGURE 7.13
Afrique's Strategy

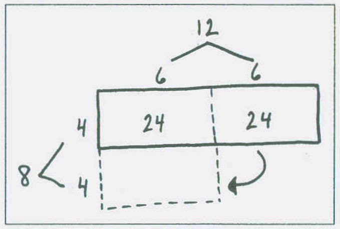
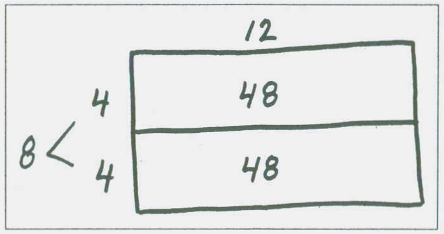


FIGURE 7.14
Verona's Strategy for 96/12



Ronnie, a fifth grader, interjects, "Hey, I just noticed this now. They both kind of doubled and doubled. Verona doubled the forty-eight and the four. Afrique doubled the forty-eight and the six."

Ronnie's insight is a "teachable moment"—an important moment to explore the connection between the doubling and halving strategy in multiplication and the reducing strategy in division. Andrea decides to focus discussion on this relationship. "Let's look at the multiplication problems for each of these division problems." She writes:

48/12 = 4
48/6 = 8
48/3 = 16
48/1.5 = 32

Then she asks the children for the multiplication problem related to each of these division problems and writes them down as well:

48/12 = 4 4 × 12 = 48
48/6 = 8 8 × 6 = 48
48/3 = 16 16 × 3 = 48
48/1.5 = 32 32 × 1.5 = 48

"What's happening in the division? What's happening in the multiplication?"

"Whew, I'm getting confused." Ray laughs.

"What's happening, Ray?" Andrea laughs too.

Ray attempts to explain. "For the answer to be the same in multiplication you had to double one number and halve the other. But with division, when the answers are the same, then both numbers have been doubled or halved."

"What about when only the divisor doubles?" Andrea probes.

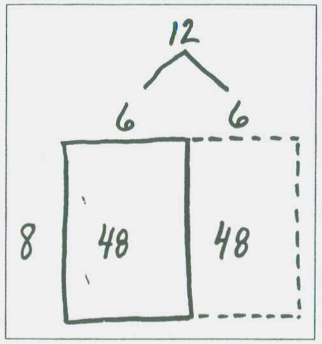


FIGURE 7.15
Afrique's Strategy for 96/12

Ray ponders, beginning to make sense of the relationship. "The answer is half," he concludes.

"And when the divisor halves?"

"The answer doubles."

Even though Ray appears to be able to explain the relationships, his understanding is probably still shaky. Many of his classmates, also, need more work with these relationships. The relationships between multiplication and division, and those involved in reducing, doubling, and halving, are very difficult for children. Andrea will need to continue this work, both with problem strings and with investigations in which children can explore changing arrays. As children represent one another's thinking using the open array, they become clearer about the relationships, which on a concrete level, with whole numbers, form the basis for later algebraic reasoning.

Using the Distributive Property of Multiplication for Division

The traditional long division algorithm is based on the distributive property of multiplication. To divide 275 by 25 using the long division algorithm, one begins by seeing how many times 25 goes into 27. The 27 is of course 27 tens, so when we say 25 goes in once, we really mean ten times. Ten times 25 is 250. The 25 that remains, divided by 25, is 1. Thus the answer is 11. What we have really done is $(25 \times 10) + (25 \times 1) = 25 \times 11$.

When we use the distributive property, we don't always necessarily have to break up the numbers in place value columns. Only the long division algorithm requires that. We could just as easily break the 275 into 200 plus 75. Two hundred divided by 25 equals 8, and 75 divided by 25 equals 3; $3 + 8 = 11$. Even nicer is 300 divided by 25 minus 25 divided by 25, or $12 - 1 = 11$.

Strings that develop the use of this strategy characterize this relationship. For example, we might start with a piece of the array, 100/25. The second problem would be another piece, such as 75/25; the third, the total: 175/25. Examples of similar strings are 150/15, 30/15, 180/15; 180/6, 36/6, 216/6; 300/6, 30/6, 270/6. The corresponding arrays are similar to the multiplication arrays (see Figure 7.16).

Putting It Together

Children can get used to being provided with the connected pieces in problem strings. For this reason, it is also important to present only one problem and have children solve it in as many different ways as they can. In this next excerpt, Andrea does just that.

"Here's a tough problem." Andrea writes 1,224/24 on the chalkboard. "How can we make it friendly?"

The students work in their math journals, calculating and drawing arrays to explain their thinking. Zoe offers to begin. "I halved both numbers to make 612 over twelve," she begins. "Then I halved again to get 306 over six. That gave me fifty-one."

"I did that, too, Zoe," Caleb says, "but I got 151."

"Is 151 times six equal to 306?" Andrea asks, encouraging Caleb to think about whether his answer is reasonable.

"No," Caleb admits. "But what did I do wrong?"

"Well, explain what you did."

Often the most powerful teaching moments come about while exploring wrong answers. Learners' strategies are always representative of their mathematical ideas, in this case about how division works. As Andrea explores with Caleb what was wrong with his strategy, his confusion about the distributive property of multiplication and its connection to division comes to the surface.

"I broke the 306 over six into one hundred over two plus one hundred over two plus one hundred over two. Then I did six divided by six, and that was one," Caleb explains, still puzzled about why this strategy didn't work.

Andrea draws a partial array of what Caleb has done (see Figure 7.17). "So what did you divide the three hundred by when you were done?" Andrea asks him to reflect on the array.

"Oh . . . only two." Caleb thinks some more. "I could do . . . let's see . . . three divided by six is one half. So three hundred divided by three is one hundred, so three hundred divided by six is fifty . . . oh, yeah. Fifty-one." Caleb has resolved his disequilibrium.

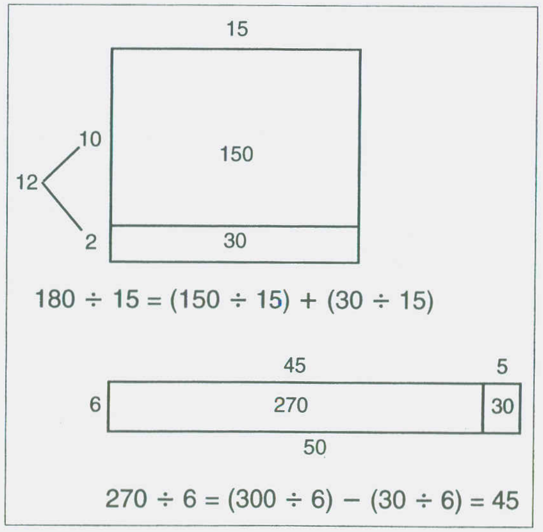


FIGURE 7.16

Andrea gives Caleb and the rest of the class a few moments to reflect and then moves on. “Any other ways? Verona?”

“I made the problem friendly by turning it into twelve hundred divided by twenty-four,” she explains. “I knew that was fifty, because twelve hundred divided by twelve is one hundred. There’s twenty-four left to be divided by twenty-four, so the answer is fifty-one.”

“Wow, that’s effective, isn’t it?” Andrea points up the efficiency of Verona’s solution. “Your way is kind of similar, isn’t it, Ron?”

Ron nods. “But I doubled. I made the problem 2,448 divided by twenty-four. I knew that answer was 102, because twenty-four hundred divided by twenty-four is one hundred, and forty-eight divided by twenty-four is two. Then I halved the answer and I got fifty-one.”

“Neat. Any other ways?”

Lena jumps in. “I did it like Zoe, but I just kept going with the halving to 153 divided by three. Then I did 150 divided by three plus three divided by three.”

Andrea’s students are flexibly composing and decomposing number just as the mathematicians did in Ann Dowker’s research study (see Chapter 6). Their constructions are also reminiscent of the Egyptian and Russian multiplication algorithms based on doubling and halving. Had these children been restricted to the long division algorithm, the first step would have been to see how many times 24 fits into 122! Not only is the value of the whole number lost, but the division is harder. And where is the creativity? Instead, they are playing with number, taking risks, constructing at the edge of their knowledge, and enjoying the aesthetics of mathematics.

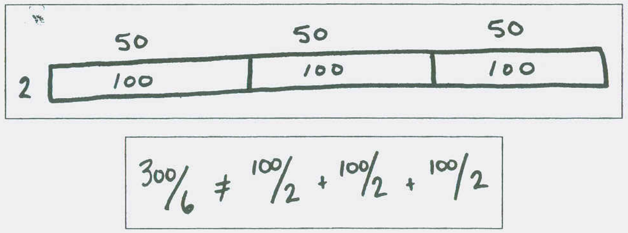


FIGURE 7.17
First Step of Caleb’s Strategy

SUMMING UP . . .

When René Descartes said, “Each problem that I solved became a rule which served afterwards to solve other problems,” he said it all. When children are given the chance to compute in their own ways, to play with relationships and operations, they see themselves as mathematicians and their understanding deepens. Such playing with numbers forms the basis for algebra and will take children a long way in being able to compute, not only efficiently but elegantly. Max Dehn envisioned the power of mathematical play, when he said: “Mathematics is the only instructional material that can be presented in an entirely undogmatic way.” Why has it taken us so long to realize it?