

3. Mental arithmetic

Learning Outcomes

By the end of this chapter you will:

- be aware of a wide range of mental calculation strategies involving the addition, subtraction, multiplication and division of whole numbers;
- understand the mathematical laws and principles which underpin mental calculation;
- appreciate that an understanding of place value and the relationships between operations is vitally important when developing mental calculation strategies.

Introduction

The status of mental methods

As discussed already in this book, being able to calculate mentally should be viewed as a vitally important component of every person's mathematical capability. Indeed, it is important that children view mental methods as always being the first resort. Whenever faced with a calculation they should always ask the question 'Can I do it mentally?' Cultivating this sort of positive attitude towards mental methods can only be achieved by teaching them in a carefully planned, structured way, which will require a considerable investment of time. Consequently, mental mathematics should not be restricted to oral-mental starters at the beginning of lessons, because these short sessions tend to focus on the practising and refining of existing skills rather than teaching new ones. A key recommendation of the Williams Review (DCSF, 2008) was a *renewed focus on oral and mental mathematics* (page 60), but this should not be interpreted solely in the form of more oral-mental starters. A more careful examination of the review reveals that it is the role and status of mental mathematics in the broadest sense that requires a greater focus as well as its relationship with pencil and paper approaches. So if you want to give mental mathematics the high status recommended by Williams, and are going to be teaching all of the techniques and approaches suggested in this chapter, you will need to set aside the main parts of a substantial number of mathematics lessons to do this – ten minutes at the start of a lesson will not be sufficient.

Mental methods should also be afforded high status beyond mathematics lessons by continually providing children with opportunities to practise and utilise their calculating skills throughout the day (see Hansen and Vaukins, 2011, for further discussion about cross-curricular mathematics). Developing mental agility across the curriculum will not only serve children well in their everyday lives, but will also equip them to access, explore and understand other aspects of mathematics more effectively, for example the use of written methods of calculation, as we

shall see in Chapter 4. It is also worth emphasising that when children start to use pencil and paper approaches this should not result in a lowering of the status of mental calculation strategies. Therefore, as children get older they should be constantly expanding the range and types of numbers they can work with mentally and continue to view mental methods as the first resort when faced with any calculation.

The importance of discussion

Williams (DCSF, 2008) highlighted the importance of high quality talk in mathematics and this is of particular relevance to the development of mental calculation strategies. It is important to encourage children to explain, share, compare and contrast the mental methods that they are using, making use of appropriate co-operative learning techniques such as 'talk partners', 'think-pair-share' and so on. Many teachers are very comfortable using these sorts of techniques to encourage speaking and listening as part of literacy lessons, or more widely across the curriculum, but not always to the same extent in mathematics. Perhaps this is because discussion-based lessons, which focus on the development of mental calculation strategies, result in children having little or nothing written in their books. The absence of written 'evidence' in an exercise book or on a worksheet does not mean the children have learned nothing, so it is not an issue to be unduly concerned about.

The key message, then, is to actively promote discussion in mathematics in the same way as in other aspects of the curriculum. One particular way of doing this is to use a child's error or inefficient mental strategy as an opportunity to introduce a more efficient one, possibly suggested by one of the other children. Not only does this develop children's understanding, it also creates an ethos in the classroom which encourages the children to 'have a go' and not to fear making a mistake or using the 'wrong' method. The ultimate aim is to build up the children's confidence as well as their ability, so that they feel comfortable using mental methods and talking about mathematics.

In order to support children's ability to talk about mathematics you should plan for the use of appropriate models and images. Resources such as 100 squares, number lines, bead strings, arrays, partitioning cards, etc. support the children's development of mental calculation strategies by offering a picture in their minds that they can use.

Another issue to be aware of is that because calculating mentally is not the same as instant recall it is important that you give children time to think and to formulate their responses. This issue was raised in the Williams Review which states that:

the review panel have observed numerous examples of undue haste on the part of practitioners during their discussions with children – in some cases even delivering the answers to their own questions before the child has had time to formulate his or her thoughts.

(DCSF, 2008, page 64)

Even if you do give children sufficient 'thinking time' you also need to be mindful of the fact that those children who work out the answers quickly, and so thrust their hands into the air, can intimidate those who are still thinking, so perhaps make use of 'no hands' approaches or something similar.

The approach adopted for a large part of this chapter is to get you to engage in a variety of mental calculations and then to use these as a vehicle to explore a wide range of strategies that can be used. This approach has the dual purpose of addressing your own subject knowledge, that is, your ability to calculate mentally, while at the same time developing your pedagogical understanding of what you will need to teach children. Although pairs of operations are considered separately in this chapter, it is important that you constantly remind children of the relationships between the four operations so that they do not view each one as a separate entity. What we do not want is a child who sees a number sentence involving subtraction to only ever think in terms of 'taking away' or counting backwards.

Mental calculation strategies for addition and subtraction

Remember, it is important to ensure that children have the prerequisite knowledge of number facts, to equip them to calculate mentally. Mental calculation strategies can start to be developed with a relatively small bank of known number facts, but it is important that this bank is expanded through regular practice, as discussed in Chapter 2.

Activity

Try to mentally work out the answer to each of the questions below. You must not write down anything other than the answer and you must not use a calculator.

When you have written down your answers, think carefully about how you tackled each question and make brief notes which explain the method you used.

1. $75 - 23$
2. $57 + 39$
3. $83 - 57$
4. $28 + 16 + 14$
5. $64 - 37 - 23$
6. $502 - 496$

Here are some possible ways that each of the questions could have been answered. Your approaches, or something similar, might be among those listed. Read the possibilities carefully and make sure you understand what is being explained in each case.

Question 1: $75 - 23$

- Visualise the traditional pencil and paper method, with 75 above 23, in columns, and the answer (52) written underneath.

- Partition the two numbers and then work out $70 - 20 = 50$ and $5 - 3 = 2$. Would it matter which calculation you do first?
- Round 23 up to 25, subtract 25 from 75 (50) and add 2 to compensate for the original rounding up.
- Subtract 20 from 75 (55) and then subtract 3 from 55, or alternatively subtract 3, then 20.

- Count back from 75 until you get to 23, probably in stages, for example 75 down to 25 (50) and then down 2 more to 23.
- Count up from 23 to 75, probably in stages, for example 23 to 30 (7), 30 to 70 (40) and 70 to 75 (5), or alternatively 23 to 73 (50) and 73 to 75 (2).

Question 2: $57 + 39$

- Visualise the traditional pencil and paper method, with 57 above 39, in columns, and the answer (96) written underneath. Would you be able to manage the 'carrying'?
- Partition the two numbers and then work out $50 + 30 = 80$ and $7 + 9 = 16$. Would it matter which calculation you do first?
- Round 39 up to 40, adding this to 57 (97) and subtract 1 to compensate for the original rounding up.
- Add 30 to 57 (87) and then add 9, or alternatively add 9, then 30.
- Add 33 to 57 (90 – but why might you do it this way?) and then add 6.

Question 3: $83 - 57$

- Visualise the traditional pencil and paper method, with 83 above 57, in columns, and the answer (26) written underneath. Would you be able to manage the 'borrowing'?
- Round 57 up to 60, subtract 60 from 83 (23) and add 3 to compensate for the original rounding up.
- Subtract 50 from 83 (33) and then subtract 7 from 33, or alternatively subtract 7, then 50.
- Count back from 83 until you get to 57, probably in stages, for example 83 down to 63 (20) and then down 6 more to 57.
- Count up from 57 to 83, probably in stages, for example 57 to 60 (3), 60 to 80 (20) and 80 to 83 (3), or alternatively 57 to 77 (20) and 77 to 83 (6).

Question 4: $28 + 16 + 14$

- Visualise the traditional pencil and paper method, with the three numbers in columns, and the answer (58) written underneath. Would you be able to manage the 'carrying'?
- Partition the three numbers and then work out $20 + 10 + 10 = 40$ and $8 + 6 + 4 = 18$. (Would you add the single-digit numbers in the order in which they appear?)

- Add the numbers as they appear from left to right, i.e. $28 + 16 = 44$ (which can be done in a variety of ways) and then $44 + 14 = 58$.
- Look along the list to find numbers that go together and so can be added quickly, i.e. $16 + 14 = 30$, and then add 28.

Question 5: $64 - 37 - 23$

- Working from left to right, calculate $64 - 37 = 27$ (which can be done in a variety of ways) and then $27 - 23 = 4$.
- Subtract 23 from 64 (41) and then subtract 37 (4). Why might it be easier to do the subtractions in this order?
- Make use of the fact that subtracting 37 and then subtracting 23 is equivalent to subtracting the sum of these two numbers, and also the fact that they can be added quickly to give 60. Finally, $64 - 60 = 4$.

Question 6: $502 - 496$

- Visualise the traditional pencil and paper method, with 502 above 496, in columns, and the answer (6) written underneath. Would you be able to manage the 'borrowing'?
- Recognise that the two numbers, despite being quite large, are very close together and so count on from 496 to 502 or count back from 502 to 496.

Efficient mental calculating strategies for addition and subtraction

There is no suggestion that the approaches listed above are equally efficient. Some are more efficient than others and it is important for you, as a teacher, to be aware of efficient and inefficient strategies.

Activity

For each of the six addition and subtraction questions presented and discussed above, identify the mental calculation strategy which is, in your opinion, the most efficient one to use. Also try to identify the one which is the most inefficient.

If you were to compare your views about efficient strategies with those of a colleague, you would not necessarily agree. This highlights an important feature of mental calculation strategies, that is, they are very personal and idiosyncratic. What works well for one person may be considered bizarre by another. Having said that, it is hoped that you do not consider the visualisation of traditional pencil and paper arithmetic to be a realistic approach to mental calculation. If you do, it is vitally important that you open your mind to the alternatives because there is certainly a more efficient approach available.

The efficient approaches utilised in answering the six questions above are based on techniques such as:

- partitioning the digits into tens and ones, or hundreds, tens and ones;
- counting up from the lower number to the higher, recognising the relationship between addition and subtraction;
- rounding to the nearest multiple of 10 and then adjusting the answer;
- looking along a list of numbers to see which ones can be combined efficiently.

In the next section you will be able to utilise these techniques again to tackle mental multiplications and divisions, and you will also be introduced to some additional techniques to add to your repertoire.

Mental calculation strategies for multiplication and division

Activity

Try to mentally work out the answer to each of the questions below. You must not write down anything other than the answer and you must not use a calculator.

When you have written down your answers, think carefully about how you tackled each question and make brief notes which explain the method you used.

1. 14×8
2. 5×39
3. $2 \times 9 \times 6 \times 5$
4. $188 \div 4$
5. 25×17
6. $340 \div 5$

Question 1: 14×8

- Visualise the traditional pencil and paper method, with 14 above 8, in columns, and the answer (112) written underneath. Would you be able to manage the 'carrying'?
- Partition 14 and then work out $10 \times 8 = 80$ and $4 \times 8 = 32$. Would it matter which calculation you do first?
- Round 8 up to 10, work out 14×10 (140) and then subtract two lots of 14 (28) to compensate for the original rounding up.
- Use repeated doubling of 14, i.e. 14, 28, 56, 112.

Question 2: 5×39

- Visualise the traditional pencil and paper method, with 39 above 5, in columns, and the answer (195) written underneath. Would you be able to manage the 'carrying'?

- Partition 39 and then work out $30 \times 5 = 150$ and $9 \times 5 = 45$ and finally add these two answers. Would it matter which calculation you do first?
- Round 39 up to 40, work out 40×5 (200) and then subtract one lot of 5 to compensate for the original rounding up.
- Multiply 39 by 10 (390) and then halve the answer to give 195.

Question 3: $2 \times 9 \times 6 \times 5$

- Multiply the numbers as they appear from left to right, i.e. $2 \times 9 = 18$, $18 \times 6 = 108$, $108 \times 5 = 540$. (How might you do these last two calculations?)
- Look along the list to find numbers that can be multiplied together quickly and also provide answers that make subsequent calculations easier, i.e. work out $2 \times 5 = 10$, then work out $9 \times 6 = 54$ and finally work out $54 \times 10 = 540$.

Question 3: $188 \div 4$

- Visualise the traditional pencil and paper method, using the division 'bracket' and with any remainders squeezed in front of the next digit.
- Partition 188 into 100, 80 and 8, then divide each of these by 4 and finally add the three answers (25, 20 and 2) together.
- Round 188 up to 200, divide 200 by 4 (50) and then subtract one quarter of the amount you rounded up (i.e. a quarter of 12).
- Use repeated halving, i.e. half of 188 is 94 and half of 94 is 47. (How might you do these two calculations?)
- Make use of the fact that $4 \times 40 = 160$. This is 28 less than 188, so 28 must be divided by 4 (7) and then added to the 40 to give 47. Essentially, this method involves partitioning 188 into 160 and 28.

Question 5: 25×17

- Visualise the traditional pencil and paper method, with 25 above 17, in columns, and the answer (425) written underneath.
- Partition 25 and then work out $20 \times 17 = 340$ and $5 \times 17 = 85$. Finally, add these two answers.
- Partition 17 and then work out $10 \times 25 = 250$ and $7 \times 25 = 175$. Finally, add these two answers.
- Round 17 up to 20, work out 20×25 (500) and then subtract three lots of 25 (75) to compensate for the original rounding up.
- Make use of the fact that every four lots of 25 is 100, so in 17 lots there will be four 100s and one lot of 25 left over.

- Recognise that multiplying by 25 is equivalent to multiplying by 100 and dividing by 4 (because every lot of 100 contains 4 lots of 25). So $17 \times 100 = 1700$ and $1700 \div 4 = 425$. (How might you divide by 4?) The two stages could be done in reverse, that is, divide 17 by 4 (4.25) and then multiply by 100 (425).

Question 6: $340 \div 5$

- Visualise the traditional pencil and paper method, using the division 'bracket' and with any remainders squeezed in front of the next digit.
- Partition 340 into 300 and 40, then divide each of these by 5 and finally add the two answers (60 and 8) together.
- Recognise that dividing by 5 is equivalent to dividing by 10 and then doubling the answer (because for every lot of 10 there are two lots of 5). So $340 \div 10 = 34$ and $34 \times 2 = 68$.

Efficient mental calculating strategies for multiplication and division

As was the case with the addition and subtraction discussed earlier in this chapter, some approaches are more efficient than others. It is important that you are able to quickly evaluate these, so as to identify efficient and inefficient mental calculation strategies.

Activity

For each of the six multiplication and division questions presented and discussed above, identify the mental calculation strategy which is, in your opinion, the most efficient one to use. Also try to identify the one which is the most inefficient.

Again, there is no suggestion that there is a 'correct' or 'best' approach, but you should have concluded that visualising traditional pencil and paper methods is not as efficient as the alternative methods. It is also hoped that you are employing mental strategies which make use of things such as near multiples of 10; the relationships between operations; partitioning into hundreds, tens and ones; and combining numbers not necessarily in the order in which they are listed. These techniques were identified earlier in relation to mental addition and subtraction, but they can also be applied to mental multiplication and division, as demonstrated above. Additionally, the following techniques have also been exemplified.

- The use of repeated doubling for multiplication by 4, 8, 16, etc. This is an example of using factors, that is, when multiplying by 8, you break it down into the factors 2, 2 and 2 and multiply by each of these. This technique can be applied in other ways, for example multiplication by 12 can be achieved by multiplying by 3 followed by 4, because 3 and 4 are a factor pair for 12.
- The use of repeated halving for division by 4, 8, 16, etc. As with repeated multiplication, this use of factors can be applied in other ways, for example division by 6 can be achieved by halving and then dividing by 3, because 2 and 3 are a factor pair for 6.

- The use of equivalent calculation combinations, for example using multiplication by 10 and halving as an equivalent to multiplication by 5, or using multiplication by 100 and division by 4 as an equivalent to multiplication by 25.

As well as employing these sorts of techniques, the mental calculations considered above require an understanding of place value as well as the commutative, distributive and associative laws of arithmetic. If the mental approaches you were using earlier have been effective and efficient, then clearly you must possess a good understanding of these laws, even if you are not aware of it! Just in case you are not fully conversant with the laws of arithmetic or the different ways of considering place value, further explanation is provided in the following sections.

In your teaching, having examined a wide range of mental techniques for addition, subtraction, multiplication and division in the sections above, it is important that you plan and teach appropriate lessons which focus on these methods, based on partitioning, near multiples of 10, repeated doubling and halving and so on, so that your children can develop high levels of mental agility.

Place value: 'column value' or 'quantity value'

An understanding of place value, like the recall of number facts, is essential for mental calculation strategies and so is required from the outset. It is important that you do not make the mistake of thinking that place value is something that is taught and practised only in Key Stage 1. As children encounter bigger numbers and different types of numbers, such as decimals, their knowledge of place value will have to be extended.

The vast majority of the mental calculation strategies discussed above are based upon a firm understanding of the concept of place value and the ability to employ the technique of partitioning. So, for example, when calculating $75 - 23$ you may have partitioned 75 into 20 and 5 and partitioned 23 into 20 and 3. Here you are using the notion of 'quantity value' because you view the 7 as 70, as opposed to viewing it as '7 tens' or '7 in the tens column', which is often the way that place value is introduced to young children. When carrying out mental calculations the 'quantity value' approach would appear to be the better way of viewing the concept of place value because it enables you to manipulate the actual numbers that the individual digits represent.

Research Focus 'Column value' and 'quantity value'

Thompson (2009) and Thompson and Bramald (2002) provide a more detailed discussion of the 'column value' and 'quantity value' aspects of place value. They also consider historical perspectives in terms of the ways that young children have traditionally been introduced to place value and how this relates to the development of mental and written calculation skills. They argue that the traditional definition of place value, which would explain the number 43 as '4 in

→

the tens column' or '4 lots of 10', is an outdated one and does not reflect the way that children actually carry out mental calculations. So, for example, when adding 43 and 52, most people would think in terms of '40 plus 50' and '3 plus 2', that is, 'quantity value' rather than 'column value'. The authors provide similar examples involving subtraction, multiplication and division and also demonstrate how 'quantity value' underpins informal pencil and paper arithmetic (which is considered in Chapter 4 of this book). It is only the traditional pencil and paper methods that utilise 'column value'.

The laws of arithmetic

The commutative law

You intuitively understand the commutative law whenever you add, subtract, multiply or divide two numbers, because you know that sometimes it is possible to switch the numbers, but on other occasions you cannot. For example, if partitioning is used to calculate $34 + 52$, the intermediate answers (80 and 6) can be added in either order, because $80 + 6 = 6 + 80 = 86$. Similarly, when calculating the product of 5 and 19 you can think of it as 5×19 or 19×5 because the answer will be the same. You can do this switching because both addition and multiplication are commutative. Subtraction and division however, are not commutative, as illustrated below.

$$12 - 4 = 8 \quad \text{whereas} \quad 4 - 12 = -8$$

$$10 \div 2 = 5 \quad \text{whereas} \quad 2 \div 10 = 0.2$$

The distributive law

If you used partitioning when you calculated 14×8 earlier, you certainly understand the distributive law. You know that 14 lots of 8 is equivalent to 10 lots of 8 plus 4 lots of 8. This is because multiplication by 8 can be distributed over the addition of 10 and 4. In formal notation this can be written as:

$$14 \times 8 = 8 \times 14 \quad (\text{the commutative law})$$

$$= 8 \times (10 + 4) \quad (\text{partitioning})$$

$$= 8 \times 10 + 8 \times 4 \quad (\text{multiplication distributed over addition})$$

$$= 80 + 32$$

$$= 112$$

Earlier you may have used near multiples of 10 to calculate 5×39 , which again demonstrates your understanding of the distributive law. You know that 39 lots of 5 is equivalent to 40 lots of 5 minus 1 lot of 5. This is because multiplication by 5 can be distributed over the subtraction of 1 from 40, as explained below.

$$\begin{aligned}
 5 \times 39 &= 5 \times (40 - 1) \\
 &= 5 \times 40 - 5 \times 1 \text{ (multiplication distributed over subtraction)} \\
 &= 200 - 5 \\
 &= 195
 \end{aligned}$$

Division can be distributed over both addition and subtraction, as illustrated by the examples below, which consider two of the ways of calculating $188 \div 4$ that were discussed earlier.

Method 1: Think of 188 as being $200 - 12$ and divide each of these by 4.

$$\begin{aligned}
 188 \div 4 &= (200 - 12) \div 4 \\
 &= 200 \div 4 - 12 \div 4 \text{ (division distributed over subtraction)} \\
 &= 50 - 3 \\
 &= 47
 \end{aligned}$$

Method 2: Think of 188 as being $160 + 28$ and divide each of these by 4.

$$\begin{aligned}
 188 \div 4 &= (160 + 28) \div 4 \\
 &= 160 \div 4 + 28 \div 4 \text{ (division distributed over addition)} \\
 &= 40 + 7 \\
 &= 47
 \end{aligned}$$

The associative law

This can be thought of as an extension of the commutative law because it relates to the order in which we combine three or more numbers. You may have demonstrated an understanding of the associative law earlier when calculating $28 + 16 + 14$ and $2 \times 9 \times 6 \times 5$. In the first question you possibly worked out $16 + 14$ first, and then added 28. You can do this because addition is associative. Similarly, in the second question you can change the order in which the numbers are combined because multiplication is associative.

Earlier, the way that you calculated $64 - 37 - 23$ probably demonstrates that you know that subtraction is not associative, as illustrated below, by using brackets to indicate the different orders in which the subtractions are being carried out.

$$\begin{aligned}
 (64 - 37) - 23 &= 27 - 23 \\
 &= 4 && \text{(which is the correct answer to the original question)} \\
 64 - (37 - 23) &= 64 - 14 \\
 &= 50 && \text{(which is *not* the correct answer to the original question)}
 \end{aligned}$$

Similarly, division is not associative, as illustrated by the following example:

$$\begin{aligned}
 (8 \div 4) \div 2 &= 2 \div 2 \\
 &= 1 \\
 \text{whereas} & \\
 8 \div (4 \div 2) &= 8 \div 2 \\
 &= 4
 \end{aligned}$$

It is not necessary for children to know these laws by name or to be taught them in a formal way. What is important, however, is that they are able to demonstrate an understanding of them when they carry out calculations, in the ways that have been discussed throughout this chapter. In essence what is required is an understanding of how the four operations can be combined, either singly or with one another, in different ways to produce alternatives, but equivalent, number sentences which result in correct answers.

Case Study

Donna is a trainee teacher in a Year 6 class on her final placement. During an ICT lesson using LOGO, some of the children are trying to write a procedure to draw a regular dodecagon and so need to work out the angle through which the turtle would turn after each forward movement. Donna is encouraging the children to use their mental skills to calculate the answer to $360 \div 12$. Here is part of the dialogue between Donna and one of the children, Ben.

Donna: So what do you think, Ben?

Ben: 18, Miss.

Donna: How did you work it out?

Ben: Well, I partitioned the 12 into 10 and 2. So 360 divided by 10 is 36 and 36 divided by 2 is 18.

Donna: Ern ... That's a good try, Ben, but not quite right. Did anyone get a different answer?

Donna knows that the answer is wrong, but she does not know why. She is therefore unable to explain to Ben the flaw in his method.

Can you see where Ben has gone wrong?

Which law of arithmetic has he tried to apply, but incorrectly?

Ben has tried to use the distributive law but inappropriately. Division by a particular number (the 'divisor') can be distributed over addition or subtraction within the number being divided (the 'dividend') as illustrated by the example earlier: →

$$188 \div 4 = (160 + 28) \div 4$$

or

$$188 \div 4 = (200 - 12) \div 4$$

However, you cannot express the divisor as the sum or difference of two numbers and then divide by each one in turn.

Division can be distributed over both addition and subtraction, but it is the dividend that must be expressed as a sum or difference, and then each part divided by the divisor.

Equipping children to calculate mentally

If we want children to use efficient mental calculation strategies, then it is important that the appropriate foundations are laid. These foundations are presented below under three broad headings.

Recall of number facts

The ability to recall number facts is a key prerequisite for efficient mental calculation. As discussed in Chapter 2, if children are relying heavily on counting techniques as a substitute for the recall of number facts, this will limit their ability to calculate mentally.

Understanding the number system

This is often referred to as having a 'feel for number' or 'number sense'. It involves having a good understanding of number patterns and relationships, including an understanding of place value. Specific examples include understanding that

- numbers ending with a zero are divisible by 10;
- all numbers ending in 0 or 5 are divisible by 5;
- all multiples of 2, 4, 6, 8, etc. are even;
- all multiples of 9 and 6 are also multiples of 3;
- alternate multiples of 3 are even and are also multiples of 6;
- a number that is divisible by 12 must be divisible by both 3 and 4.

It is also important to understand the nature of the four number operations, including their commutative, distributive and associative properties, as well as the way that they relate to one another. Specific examples of this include understanding:

- that when adding, increasing one number by a fixed amount and decreasing the other by the same amount will not affect the answer, for example $29 + 36 = 30 + 35$ (add 1 to one number and subtract 1 from the other);

- that when subtracting, you can increase or decrease both numbers by the same amount and it will not affect the answer, for example $81 - 24 = 80 - 23$ (subtract 1 from both numbers);
- that when multiplying, you can multiply one number by a fixed amount and divide the other by the same amount and it will not affect the answer, for example $15 \times 18 = 3 \times 90$ (divide one number by 5 and multiply the other by 5);

- that when dividing, you can multiply or divide both numbers by the same amount and it will not affect the answer, for example $390 \div 15 = 39 \div 1.5 = 78 \div 3$ (divide both by 10, then multiply both by 2);
- the inverse relationship between addition and subtraction, so that, for example, a subtraction can be tackled by counting on from the lower number to the higher;
- the inverse relationship between multiplication and division, so that, for example, $480 \div 40$ can be thought of as 'What do I multiply 40 by to get 480?';
- multiplication as repeated addition;
- division as repeated subtraction;
- the 'oddness' or 'evenness' relationships when numbers are added, subtracted, multiplied and divided; e.g. if you multiply two odd numbers the product will be odd;
- the effects of adding, subtracting, multiplying and dividing by zero and 1.

Mental arithmetic techniques

Children need to be able to utilise their recall of number facts, in conjunction with their understanding of the number system, to employ a wide range of techniques to carry out efficient mental calculations. These are the sorts of techniques that were discussed earlier in this chapter, when you tackled six additions and subtractions and then six multiplications and divisions. Some children who can recall number facts and have a good 'feel for number' will be able to devise these techniques themselves, but the majority of children are not capable of this and so they have to be taught in a carefully planned, structured way.

Research Focus: A model of mental calculation

Thompson (2010) provides a historical perspective on the teaching of mental calculation and also presents a summary of key research in this area. He concludes by presenting a *model of mental calculation*, which comprises *facts* (for example addition and subtraction facts to 20, multiplication tables and corresponding division facts), *understandings* (what Thompson refers to as *number sense* and including the sorts of things outlined in the section above) and *skills* (for example counting on from the higher number when adding, rather than counting all of the objects). These three components of Thompson's model of mental calculation correspond closely to the three subheadings used above. Additionally, Thompson's

→

model identifies attitudes as the fourth component of his model, that is, children's confidence in their ability to use mental strategies, which Thompson describes as *an important but neglected ingredient* (page 170). He states that a change of attitude to mental calculation is needed so that children are prepared to *have a go* and not simply say *I can't remember the method so I cannot solve the problem*.

Implications for your teaching

As stated above, it is important that mental calculation strategies are taught in a planned, structured way. To assist you with this, the following general guidelines are offered, some of which have been suggested already in this chapter.

The importance of counting

Counting activities form the basis of early calculation and help children to develop a 'feel for number' but, like place value, these should not be restricted to Key Stage 1. Older children can develop their understanding of the number system, and therefore their ability to calculate mentally, by engaging in activities such as: counting forwards and backwards in multiples of 3, 4, 5, etc; counting in multiples but not starting at an exact multiple; counting involving repeated doubling or halving; counting involving sequences such as square numbers; counting forwards and backwards in larger steps, to encourage mental calculation. So, for example, you could ask children in upper Key Stage 2 to chant the multiples of 17 – '17, 34, 51, 68, 85 ...' – and discuss the strategies they use to quickly work out each one. Discussion-based activities such as this result in nothing being written down, but nevertheless can result in much learning taking place. By employing co-operative learning techniques, like those suggested at the beginning of this chapter, it is also possible to involve all children in the discussions. There will also be opportunities to effectively utilise models and images, such as number lines, counting sticks, 100 squares, etc.

Using and applying mental skills

Like any other mathematical technique or skill, the whole point of developing and mastering efficient mental methods is to use them – you should not be teaching these things just for the sake of it. It is therefore vitally important that you provide appropriate opportunities for children to use and apply their mental skills in mathematics through problem-solving activities and open-ended, investigative tasks. The use and application of these skills should also extend right across the curriculum and throughout children's everyday lives.

Mental arithmetic – always the first resort

The status of mental mathematics has already been considered at the beginning of this chapter, but it is appropriate to reiterate this again here. Children of all ages as well as adults, should be constantly striving to develop and extend their mental capabilities, because these are essential

skills for life. They never become redundant and should never be abandoned when pencil and paper methods are introduced, or when an electronic calculator becomes available; indeed they facilitate the effective development of these alternative approaches, as we shall see in Chapters 4 and 7. Children should be proud of their ability to mentally juggle with large numbers, negative numbers, fractions, decimals, percentages and anything else that their everyday lives throw up.

Learning Outcomes Review

You should now be aware of a wide range of strategies that can be utilised when carrying out mental arithmetic, as well as appreciate the vitally important role they play in equipping us all, children and adults, for our everyday lives.

However, having 'awareness' is not sufficient – you need to be able to put these strategies into practice so that you are able to mentally juggle with numbers and calculate efficiently. If mental arithmetic is not one of your strengths (possibly because it was not given sufficient emphasis when you were a child at school) then you must spend time practising and developing your own skills, so that you can be a role model for the children you teach.

As well as knowing a range of strategies, you should understand the laws of arithmetic, the role of place value and the interrelated nature of the four arithmetical operations, and also appreciate how these underpin mental arithmetic.

In teaching mental arithmetic you should appreciate that children need to be secure in their knowledge of number facts, should have a good understanding of the number system and should be confident when working with numbers. Finally, they need to be taught all of the calculating techniques in a structured way because the majority of children are not capable of developing these for themselves.

Self assessment questions

1. If, when calculating 13×25 , you partition 13 into 10 and 3 and then multiply each of these by 25, which of the laws of arithmetic are you basing your method on?
 - a. $135 \div 5$
 - b. $38 + 26 + 26$
 - c. $5004 - 1947$
 - d. $33 \times 7 \times 3$
2. If, when calculating $17 \times 5 \times 2$, you work out 5×2 first and then multiply the answer by 17, which of the laws of arithmetic are you basing your method on?
3. Work out the answer to each of the following questions by using an efficient mental method. Make a note of the answer and the method you use.
 - a. $135 \div 5$
 - b. $38 + 26 + 26$
 - c. $5004 - 1947$
 - d. $33 \times 7 \times 3$

Further Reading

- DFE (2010) *Teaching Children to Calculate Mentally*. London: DfE Publications.
- Hansen, A. and Vaukins, D. (2011) *Primary Mathematics across the Curriculum*. London: Learning Matters. Read Chapter 2 ('Number') although other sections of the book will also be of general interest.
- QCA (1999) *Teaching Mental Calculation Strategies: Guidance for teachers at Key Stages 1 and 2*. Sudbury: QCA Publications.

References

- DCSF (2008) *Independent Review of Mathematics Teaching in Early Years Settings and Primary Schools* (The Williams Review). Notttingham: DCSF Publications.
- Thompson, I. (2009) 'Place value?' *Mathematics Teaching*, 215: 4–5.
- Thompson, I. (2010) Getting your head around mental calculation, in Thompson, I. (ed.) *Issues in Teaching Numeracy in Primary Schools*. Maidenhead: Open University Press.
- Thompson, I. and Brannald, R. (2002) *An investigation of the relationship between young children's understanding of place value and their competence at mental addition*. Final report submitted to the Nuffield Foundation. Newcastle: Department of Education, University of Newcastle upon Tyne.

4. The development of pencil and paper arithmetic

Learning Outcomes

By the end of this chapter you will:

- appreciate that pencil and paper arithmetic evolves naturally from the mental calculation strategies considered in Chapter 3;
- be aware of a range of informal pencil and paper approaches, covering all four operations;
- appreciate the role and status of the informal methods discussed in this chapter;
- understand progression in the development of pencil and paper arithmetic, eventually working towards the introduction of the traditional algorithms presented in Chapter 5.

Introduction

Before the advent of the National Numeracy Strategy in 1999 it was common practice to introduce children to the traditional pencil and paper methods for addition, subtraction, multiplication and division as early as possible. Typically, in the case of addition and subtraction, this was at the age of 6 or 7, that is, as soon as children were able to mentally add and subtract two single-digit numbers. However, research has shown that this was not effective, with many children failing to understand what they were doing and often getting incorrect answers. The 1999 *Framework for Teaching Mathematics* (DFEE, 1999) provided clear lines of progression in the development of pencil and paper arithmetic during Key Stage 2 by presenting informal, expanded written methods based on the mental strategies that should have already been taught. The traditional pencil and paper methods were presented as being the final stage in this progression, instead of the initial one as had previously been the case. Even the most recent guidance from the National Strategies (2011) continued to support this model of progression in children's arithmetic, although the guidance did emphasise that the informal, expanded pencil and paper methods should be viewed as 'staging posts' on the way to more compact, efficient methods, rather than being end points in themselves.

In its deliberations over the development of the new mathematics National Curriculum for England, the Department for Education has expressed similar views. It has stated that:

- *Children must be fluent in applying quick, efficient written methods of calculation.*
- *Children need to be moved on rapidly to use appropriate efficient written methods,*