Milan Hejný
Faculty of Education, Charles University in Prague,
Czech Republic

Scheme – oriented educational strategy in mathematic

Scheme is understood as a memory structure that incorporates clusters of information relevant for comprehension. It gets embedded in a person's mind by a repeated "stay" in a certain kind of environment (one's house, school, shopping centre). Scheme–oriented mathematical education is described and illustrated on a primary level. This paper surveys the experience with the implementation of this teaching method in teacher's training.

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1. Introduction

The aim of the contribution is to analyse and discuss one cognitive phenomenon which, to our opinion, might be used to improve contemporary education of mathematics (not only) in elementary schools.

Contemporary educational strategy of mathematics in the majority of our (Czech and Slovak) elementary classes is topic oriented. It means that each time, the whole period of mathematical lesson is focused on a particular topic: counting, sharing, measuring, etc. The alternative educational strategy presented in this paper is scheme oriented. It means that mathematical lesson is focused on solving problems, in which more mathematical schemes are addressed at the same time. A set of what we call mathematical environment is an educational tool for such an approach.

2. Scheme

When someone asks you about the number of windows or lamps in your flat or house, probably you will not be able to give an immediate answer. However, after a little while you will answer the question with absolute certainty. You will imagine yourself walking from one room to another and counting the objects that you were asked about. Both of the required pieces of information and many other data about your dwelling is embedded in your consciousness, as a part of the scheme of your flat. We use schemes to recognize not only our dwellings, but also our village, our relatives, interpersonal relationships at our workplace, etc.

Specialized literature gives various connotations of the term 'schemes'. The following quote by R.J. Gerrig provides a rather loose definition that serves our purposes. "Theorists have coined the term schemes to refer to the memory structure that incorporate clusters of information relevant to comprehension… A primary insight to scheme theories is that we do not simply have isolated facts in memory. Information is gathered together in meaningful functional units." (Gerrig, 1991, pp. 244–245).

A scheme-oriented education is based on creating two kinds of schemes:

semantic schemes rooted in everyday life experiences of a pupil and *structural* schemes which are 'pure mathematical' and have no direct linkage to pupil's life experience.

Structural schemes of early mathematics are created within different semantic schemes and after introducing a structural language of ciphers, they start to shake off this semantic supervision.

As an example, let us consider the concept of 'number 3' as one of the ten basic elements of the early mathematic structural scheme. This mathematical concept originates from the semantic scheme of rhymes, creating the ability to produce the rhythm, which synchronizes words and movements. A child's speech *one, two, three* is accompanied by handling obejcts. After the performance, its last word *three* must be repeated to point at the product of the process of counting: the set of three objects. Once the synchronization is created, a child is able to count objects. Both abilities – handling objects and synchronization – are nested in everyday life experiences. Word *three* in both its appearances starts to create a concept of 'three objects' as the first pre-structural concept of the concept 'number 3'. The first word 'three' brings the processual and the second 'three' the conceptual understanding.

The described pre-structural concept of 'number 3' is not completed yet. So far it is supported only by one semantical scheme and three others have to be added: number as an address, number as an operator of comparison and number as an operator of change (see below).

3. Semantic schemes breeding up the early arithmetic structural scheme

As mentioned above, there are four different semantic schemes in which a number appears: status S (number, magnitude), address A (in terms of place or time; the temporal address can be either linear of cyclical), the operator of change Ch and the operator of comparison Co. The symbol O will stand for the operator, when there is no need to specify its particular type.

In some cases there is no sharp boundary between these schemes. Take for example this situation: Ann (who stays at the second floor) has to go three floors up in order to see Betty (who stays at the fifth floor). Here number 3 can be regarded either as an operator of change (Ann moves) or as an operator of comparison (Betty stays 3 floors above Ann).

Numbers are the soil of an early arithmetic scheme. The core of it is an operation – addition and subtraction. Here the variety of semantic types comprise at least eight issues:

S + S = S	3 female and 5 male pupils, 8 pupils in total.
S – S = S	If 5 out of 8 pupils are boys, then the remaining 3 are girls.
S ± Co = S	E has 3 pets. F has 1 pet more/fewer than E. Thus F has 4/2 pets.
A ± Co = A	J. is 8 years old. R. is 1 year older/younger. R. is 9/7 years old.
S ± Ch = S	Eve had 5 €. Today she received/lost 2 €. Now she has 7/3 €.
A ± Ch = A	Cid used to live on the 5th floor. He moved 2 floors up/down. Now he lives on 7^{th} /3^{rd} floor.
Co ± Co = = Co	Eva read 5 pages more than Fay, who read 2 pages more/fewer than Guy. Eva read 7/3 pages more than Guy.
Ch ± Ch = = Ch	The number of bus-passengers increased by 7 persons at the first stop. At the second stop it increased/decreased by 5 persons. At these two stops the number increased by 12/2 bus-passengers.

Table 1

The key semantic model, mastery of which is the decisive step towards understanding Early Arithmetic scheme, can be written as ±O ± O = O. Our longterm experience substantiated by the experimental research of Ruppeldtová (2003), clearly indicate that the problems of using only operators are among the most demanding problems for pupils. We would like to know why operators are so demanding?

Commentary 1. The answer to the given question is rooted in different perceptions of statuses and addresses on one hand, and the operators on the other. The status and address are both enclosed data. Information such as "there are 5 chairs around the table" does not generate any further questions concerning numbers.

The operator is, by contrast, an example of an open data. The information "there are two chairs fewer" provokes the question such as: 'what was the original number of chairs?' and 'how many chairs are there now?' These two numbers are *virtually* present in the operator of change. The accuracy of the above thesis is confirmed by the behaviour of pupils who are assigned to such operator problems. When given such a problem, they keep asking for virtual data and for explanations as to how to deal with them. These pupils clearly did not have enough experience with numerical situations that feature exclusively the operator of change. That is why the current situation might be improved by incorporating operator tasks already in first-grade primary school curricula. In order to achieve this goal, we elaborated several environments. Three of them are presented in this paper.

4. 'Walk' environment

The teacher (and later one of the pupils) gives an order and another pupil(s) walks accordingly to it. Sample commands: 1. Three steps forward, go! 2. Two steps, then one step, forward, go! 3. Three steps forward, then two steps backwards, then one step forward, go! To keep steps of pupils equal, there is a set of about a dozen marks on the floor of the class. After this warm-up stage, the addition is introduced by the following scene: Two pupils, C and D, are standing side by side. Pupil C receives the following command: Three steps forward, then two steps forward, go! Pupil D receives the command: Five steps forward, go! Both pupils, C and D, eventually end up standing side by side again. The entire scene is accompanied by words and body movements, and can be classified as a walk representation of the addition 2 + 3 = 5.

The problem originates by concealing one of the three numbers. The given situation, therefore, leads to three problems: 2 + 3 = ?, 2 + ? = 5, ? + 3 = 5. The concealed number here has been replaced by a question mark. In the class scenario it is replaced by the word 'what?'; e.g. problem written here as 2 + ? = 5 will be presented as

Problem 1. Pupils C and D are standing side by side. Pupil C goes 5 steps forward. Then a teacher says 'Pupil D two steps forward, then *what?* steps forward, go!'.

The class already knows that it is necessary to replace the word 'what?' by a suitable number. In the given case the number is 'three'.

'Walk' environment brings a natural possibility to introduce the pre-concept of negative numbers (which is impossible within the environments dealing only with a status). Negative numbers are represented by backward steps. The experiment proved that even firstgraders can easily solve the problem 2 – 3 = ? Pupil C receives the following command: Two steps forward, then three steps backward, go! Few pupils immediately and the whole class after a while found the solution as a command for pupil D: one step backward. In such a way a concept of negative number starts within the backward movement.

It is necessary to stress that on this stage, there is no numerical notation for negative numbers. Symbols like '-1' will be introduced later, not before the fourth grade. At that time, in each of our experimental classes, pupils used a sign minus as a natural description for both: addresses of places/years below the zero and operators of change in decreasing directions.

Commentary 2. The 'Walk' environment allows pupils to build their semantic schemes, from which four Early Arithmetic fundamental sub-schemes emerge: number ordering, addition, subtraction within natural numbers and pre-concept of negative number. The most important in this environment is a great support for understanding of addition and subtraction of operators, particularly the operator of change.

5. 'Footprint' environment

So far we dealed with short commands only. When a longer command with five or even more numbers appears, it will be difficult for a pupil to remember it. Thus, there is a need to find a way how to record a long command. Pupils start to create their own recording systems using fingers, dots, lines,… Finally one or more of pupils finds an arrow as a suitable tool for recording steps. A teacher now can take this pupils' discovery as a common language for describing commands and walk performances. It is important that no authority such as a teacher or a textbook actually brought this new language. Pupils found it themselves and therefore it is their own language. In such a way the Footprint environment is introduced[1] .

The arrow representation of the addition 2 + 3 = 5 is given by Figure 1.

$$\boxed{\rightarrow\rightarrow|\rightarrow\rightarrow\rightarrow} = \boxed{\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow}$$

Figure 1

On Figures 2a, 2b, 2c there is an arrow representation of tasks 2 + 3 = ?, 2 + ? = 5, and ? + 3 = = 5 respectively.

$$\boxed{\rightarrow\rightarrow|\rightarrow\rightarrow\rightarrow} = \boxed{},\quad \boxed{\rightarrow\rightarrow|} = \boxed{\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow},\quad \boxed{|\rightarrow\rightarrow\rightarrow} = \boxed{\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow}$$

Figure 2a Figure 2b Figure 2c

There are two substantial differences between environments: Walk and Footprints. The first one is due to the fact that the Walk is ephemeral, while the Footprints is permanent. Words and steps will fade away, but Figure 1 will remain.

The second difference resides in the fact that the permanent language allows to create more demanding tasks than the language of word commands. This can be illustrated by one problem dedicated for fourgraders.

Problem 2. Fill in the three empty boxes with six arrows to fulfill both equations:

$$\boxed{|\leftarrow\leftarrow\leftarrow} = \boxed{\rightarrow|} = \boxed{}. \qquad (1)$$

Remark. Only arrows of the same direction are allowed in each box.

[1]J. Slezaková (2008) performed a number of experiments with the sole aim of finding appropriate graphemes for this model. In the end, arrow symbols were chosen as the most appropriate for children of 6 to 8 years of age.

Commentary 3. Having translated problem 2 into algebraic notation, it can be written as the system of three equations:

$x - 3 = y + 1 = z, |x| + |y| + |z| = 6;$

number x is positive if arrows in the first empty box in (1) are oriented right (→) and negative if these are oriented left (←). The same procedure is valid for letters x and z.

Within this environment even such a difficult rule

minus out of minus makes plus (2)

can be presented as a walking performace by means of the command 'turn about' abbreviated by TA. For example the expression 3 – (2 – 4) can be produced as shown in arrow languge:

→ → → | TA | → → | ← ← ← ← | TA

Figure 3

Our experience with this interpretation of the rule is very positive. Many of our pre-service teachers, future elementary teachers declare this performance to be ‚the proof of the rule'.

Commentary 4. While solving various Walk & Footprints problems, a pupil gets familiar with this double-environment and develops his/her mathematical understandig in different areas: ordering, addition, and subtraction of whole numbers; later on also solving system of equations, pre-concept of the absolute value of a number and even several ideas from probability and statistics.

6. A 'bus' environment

The bus route is marked by several (shall we say five) stops in the classroom, which we shall label A, B, C, D, and E. The stops are at particular places within the classroom, e.g. the teacher's desk, a washbasin, map, whiteboard, wardrobe, the piano, … A cardboard box stands for the bus and plastic bottles stand for the passengers. The bus departs from the initial stop A and ends up in the terminus E and anyone can get off and get on each stop. The decision-making is done by the pupils who act as conductors at individual stops. All the pupils see how the passengers are getting on and off, but only the driver can see the inside of the bus (the box); the driver is the pupil who is carrying the box. When the bus has reached the terminus, the teacher asks the pupils how many passengers they think there are in the bus. Each pupil writes his/her tip into a table and then checks it by looking inside the box.

The pupils first try to remember the number of passengers, later they start to keep written records. After the fifth or sixth round of the game, the teacher asks whether anybody remembers how many people got off at stop B. The teacher asks such questions during every subsequent performance, which forces the pupils to invent a more resourceful way of recording the entire process.

Story. In one experimental class, where each stop had a distinct colour, the teacher, after a tenth round asked: "What happened at the green stop? Did the number of passengers increase or decrease? By how many?" Only several pupils understood those difficult questions. One pupil immediately gave a correct answer. Then he explained to his classmates the secret of his solution.

Prior to the performance, the boy drew up 5 oval shapes in five different colours which matched the colours of the bus stops. Each oval stood for a bus standing at the appropriate stop. Then, using arrows, he recorded the performance. Two arrows directing the green oval represented two passengers entering the bus at the green stop and four arrows directing out of the oval represented four passengers getting out of the bus. Having used this record, the boy immediately saw that at the green stop the number of passengers decreased by two.

After several performances, some pupil came up with a method of table- recording. On request of the teacher, the discoverer showed the record to the class and the teacher started writing his/her own performance records on the blackboard. The discovery was not made in all experimental classrooms, in some of these the teacher had to clue the pupils and gave them the table record. The pupils eventually used the record found in the upper half of table 2. The table shows us that 2 passengers got off at stop B, while 3 got on; on the stop C one passenger got off and 4 got on. After a month, the table was complemented with a row entitled "go" which stands for the number of passengers on the bus between individual stops. E.g. Table 2 indicates that the highest number of people – 7 – went from stop C to stop D.

	A	B	C	D	E
out		2	1	4	5
in	3	3	4	2	
go	3	4	7	5	

Table 2

The teacher gives the pupils tasks that reside in blotting out some of the numbers. It can be illustrated by a rather demanding task assigned to 4th-graders.

<u>Problem 3</u>. Design a performance table if you know that the same number of people got on at each of the A, B, C, D stops. 5 people got off at stop D and the maximum of 5 people and the minimum of one person got off on the subsequent stops.

<u>Commentary 5</u>.The Bus environment was tested by seven different teachers in 4 first-grade and 3 second-grade classrooms. The environment was given a very warm welcome, especially by pupils. This influenced even those teachers who did not trust the environment at first. The environment uses the experience of pupils with bus route.

After illustrating three semantical environments, we turn our attention back to the starting concept of our research, to the scheme.

7. Theory of generic models – a tool for understanding mathematical scheme

Remark. From now on, under the term 'scheme' we mean 'mathematical scheme'.

In chapter 2 we gave Gerrig's definition of the concept scheme. Then, we gave several illustrations of mathematical schemes. However, the concept still remains in a theoretical level. So far, we have no clear idea how to use this concept in our educational praxis. Namely we do not know

a) How to evaluate the quality of particular mathematical scheme
in a given pupil's mind? and (3)

b) How to help this pupil to overcome possible developmental obstacles?

The goal of this chapter is to find such a tool. To do this, we will use our theory of generic models, which we briefly (i.e. without illustrations) describe[2] here.

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[2]The theory designed by the author's father, Vít Hejný was first published in 1977 in the Slovak language. Its first English presentation is found in the article Hejný (1988), and its current version is in the paper Hejný, Littler (2006).

Our model of the process of gaining knowledge is based on stages. It starts with motivation and its cores are two mental lifts: the first (generalisation) leads from a concrete knowledge to a generic knowledge and the second (abstraction) from a generic to an abstract knowledge. The permanent part of the knowledge gaining process is crystallisation – inserting new knowledge into the already existing mathematical structure.

The whole process can be depicted in the following scheme consisting of two consequent levels:

motivation → isolated models → *generalisation* → generic model(s)

generic model(s) → *abstraction* → abstract knowledge → crystallisation

As we see, the generic model, the pivot between experiences and abstract knowledge, plays a decisive role.

<u>Motivation</u>. We see motivation as the tension which occurs in a person's mind as a result of the discrepancy between the existing and desired states of knowledge. The discrepancy comes from the difference between 'I do not know' and 'I need to know', or 'I cannot do that' and 'I want to be able to do that'. Sometimes this discrepancy comes from other needs too.

<u>Isolated models</u>. First experiences of a new piece of knowledge come into mind gradually and have a long-term perspective. For instance, the concepts of fraction, negative number, straight line, congruency or limit develop over many years at a preparatory level. For more complex knowledge, the stage of isolated models can be divided into four sub-stages:

1. The first concrete experience – the first isolated model appears and this is a *source* of new knowledge.
2. A gradual 'collecting' of more isolated models, which at this stage are separate.
3. Some models begin to refer to each other and create a *group*. The feeling develops that these models are 'the same, in a sense.
4. Finding out the reason for the 'sameness', or even better, the correspondence between any two models. These models create a *community.*

The above sub-stages can be useful for us when we investigate how a new idea gradually develops in a pupil's mind. It often happens that a new sub-stage, not presented here, appears and that one of those presented does not appear at all.

The stage of isolated models ends with the creation of the community of isolated models. In the future, other isolated models will come to a pupil's mind, but they will not influence the birth of the generic model. They will only differentiate more detail in it.

<u>Generalisation and generic model(s)</u>. In the scheme of the process of gaining knowledge, the generic model is placed over the isolated models indicating its greater universality. The generic model is created from the community of its isolated models and has two basic relationships to this community:

1. it denotes both the core of this *community* and the core of *relationships* between individual models and
2. it is an example or representative of all its isolated models.

The first relationship denotes the construction of the generic model; the second denotes the way the model works.

<u>Abstraction and abstract knowledge</u>. The generic model remains an object representative and does not allow for a higher level of structuring acquired knowledge. Therefore, the next step of knowledge development must be abstraction, i.e. disconnection from an object characteristic of a generic model. This shift is accompanied by a change of language and an object representative is exchanged for a symbolic representative. The symbolic representative brings about higher abstract understanding of the knowledge or knowledge area in question than the previous

object representative does. This process is intellectually demanding and requires a lot of time and effort from a pupil. The abstract knowledge is only rarely a consequence of AHA-effect, i.e. a sudden sight of truth. A majority of abstraction processes run in small stages. Creating abstract knowledge is based on the assumption that the symbolic representative is autonomously constructed or at least interiorized by an individual. If the symbolic representative is implemented in a pupil's mind from the outside in a ready-made form it usually only stays on a memory level as 'the knowledge without understanding'.

Crystallisation. After its entrance into the cognitive structure, a new piece of knowledge begins to look for relationships with the existing knowledge. When it discovers disharmonies, the need arises to remove them by adapting the new knowledge to the previous knowledge and, at the same time, to change the previous knowledge to match the new knowledge.

The above description of crystallisation is imprecise in two aspects: first, it suggests the image that crystallisation only begins when the abstract knowledge has been constructed. Second, it supposes that the only thing that is added to the cognitive structure and takes part in the process of crystallisation is the abstract piece of knowledge. Neither is true. Each new mental step, which plays a role in creating the new abstract knowledge, immediately becomes a part of the whole cognitive structure and plays a role in crystallisation. None of the pieces of knowledge which a pupil constructs has a final form and each is being polished, changed and broadened all the time. This permanent development of knowledge is a typical sign of the quality of non-mechanical knowledge.

8. Cognitive mechanism of the birth and the rise of mathematical scheme

Now we are prepared to answer questions (3). In brief we can say that

a) the quality of particular mathematical scheme in a given pupil's mind can be evaluated accordingly to the set of its generic models and a web by which these models are connected;
b) the most frequent developmental obstacles originate from the lack of generic models and their connections; thus the way of overcoming these obstacles is to build these missing models and connections.

In more details we describe the birth of the scheme and its internal organization.

Firstly, we clarify the birth of scheme. Isolated models and clusters of these models provide a breeding ground for a scheme. A scheme only appears with the origination of the first generic model. A child may discover that the total of 2 footballs and 3 footballs equals 5 footballs, the total sum of 2 and 3 dolls is 5 dolls, but s/he has not yet developed a scheme for adding small numbers. This scheme is only developed once the child has discovered that these calculations can be done by counting on fingers, which thus become a generic model for adding small numbers.

Secondly, we underline that scheme is a dynamic organisation of heterogeneous elements. The word *organisation* emphasises the fact that it is not just a set of elements, but also a set of bonds between these elements. The adjective *dynamic* refers to both short- and long-term mutability of the set of elements and of the entire organisation. Schemes may be either more stable or more flexible. Some flexible schemes originate by the amalgamation of smaller schemes. E.g. the scheme of the term "rational number" was created by amalgamating the schemes of the terms "fraction" and "negative number". The dynamism of a scheme is shaped by an internal conflict following the introduction of a new isolated model: a 1st-grader discovers that one half is a number, or a 4th-grader realises that a quadrilateral can be non-convex, or an 8th grader finds out that there can be a triangle with indefinitely large circumference and indefinitely small area.

9. Conclusions

As stated in the Introduction, one of the goals of this paper is to prepare our future activity while working with teachers. The key role in this work will be played by schemes, isolated and generic models. Here, these concepts are illustrated mostly in the area of arithmetic. However, in this work we will deal with geometry as well. As a base for the geometrical activity we will use ideas described in Swoboda (2006), Jirotková (2007), and Hejný, Jirotková (2006).

The concept of the scheme is elaborated in several theories. For example in a famous study of Gray, Tall (1994), in which the concept of procept is introduced. We read:

"The ambiguity of notation allows the successful thinker the flexibility in thought to move between the process to carry out a mathematical task and the concept to be mentally manipulated as a part of the *wider mental scheme*. Symbolism that inherently represents the amalgam of process/concept ambiguity we call a 'procept'" … (p. 116).

The concept of the scheme is also incorporated in the APOS theory. It presupposes "… that mathematical knowledge consists in an individual's tendency to deal with perceived mathematical problem situations by constructing mental *actions*, *processes*, and *objects* and organizing them in *schemes* to make sense of the situations and solve the problems. … Finally, a *scheme* for a certain mathematical concept is an individual's collection of actions, processes, objects, and other schemes which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept" (Dubinsky, McDonald, 1999).

Interpretations of the scheme in procept theory and in APOS theory are similar to our interpretation. The comparison of these theories can be found in Hejný (in print)

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