Vaje 27. 12.

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Ekstremi funkcij več spremenljivk

(*) Kako poiskati ekstreme?

Naj bo $D \subset \mathbb{R}^n$ domena, $f: D \to \mathbb{R}$ gladka funkcija.

(*) ↳ Najprej poiščemo stacionarne točke:

- $a \in D$ je stacionarna, če je $\nabla f(a) = 0$, tj.

$$\frac{\partial f}{\partial x_i}(a) = 0 \quad , \quad i = 1, \dots, n$$

- naj bo a stacionarna

$\rightsquigarrow$ pogledamo $H_f(a) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(a) \right)_{i,j=1}^{n}$

↳ poiščemo lastne vrednosti $\lambda_1, \dots, \lambda_n$

↳ če so vse $\lambda_i > 0 \Rightarrow a$ je minimum

↳ če so vse $\lambda_i < 0 \Rightarrow a$ je maksimum

V primeru $f: \mathbb{R}^2 \to \mathbb{R}$:

$H_f(a)$ je 2×2 - matrika, kot vemo je tudi simetrična:

$$H_f(a) = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$$

↳ $\det H_f(a) = xz \underbrace{- y^2}_{< 0} = \underbrace{\lambda_1 \lambda_2}$

↳ če je $\underbrace{\lambda_1 \lambda_2 > 0} \Rightarrow a$ ekstrem

$\Rightarrow xz > 0$

↳ $x > 0 \Rightarrow z > 0 \rightsquigarrow \lambda_1, \lambda_2 > 0$

↳ $x < 0 \Rightarrow z < 0 \rightsquigarrow \lambda_1, \lambda_2 < 0$

↳ Če imamo $f: \mathbb{R}^2 \to \mathbb{R}$, $a \in \mathbb{R}^2$ stacionarna točka:

- $\det H_f(a) > 0$ in $\frac{\partial^2 f}{\partial x^2}(a) > 0$

$\Rightarrow a$ je minimum (lokalni)

- $\det H_f(a) > 0$ in $\frac{\partial^2 f}{\partial x^2}(a) < 0$

$\Rightarrow$ a je maksimum

. $\det H_f(a) < 0 \Rightarrow$ a ni ekstrem

N1: Preveri, da ima funkcija

$$f(x,y) = (1+e^y)\sin x - ye^y$$

neskončno mnogo maksimumov in nobenega minimuma.

R1:

$$\frac{\partial f}{\partial x} = (1+e^y)\cos x$$

$$\frac{\partial f}{\partial y} = e^y \sin x - (1\cdot e^y + ye^y) =$$

$$= e^y(\sin x - 1 - y)$$

V katerih točkah velja $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$?

i) $(1+e^y)\cos x = 0$

$1+e^y > 0$, $\cos x = 0$

$\hookrightarrow x = \frac{\pi}{2} + k\pi \; ; k \in \mathbb{Z}$

ii) $\cancel{e^y}(\sin x - 1 - y) = 0$

$y = \sin x - 1$

$\hookrightarrow$ 1) $x = \frac{\pi}{2} + 2k\pi$: $\sin x = 1$

$\hookrightarrow y = 0 \rightsquigarrow (\frac{\pi}{2} + 2k\pi, 0)$

(to so stacionarne točke)

2) $x = -\frac{\pi}{2} + 2k\pi$: $\sin x = -1$

$y = -1 - 1 = -2$

$\hookrightarrow (-\frac{\pi}{2} + 2k\pi, -2)$

(to so tudi stacionarne to.)

$\hookrightarrow$ Sedaj izračunamo odvode drugega reda in določimo Hessejevo matriko:

$$\frac{\partial f}{\partial x} = (1+e^y)\cos x$$

$$\hookrightarrow \frac{\partial^2 f}{\partial x^2} = -(1+e^y)\sin x$$

$$\hookrightarrow \frac{\partial^2 f}{\partial x \partial y} = e^y \cos x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(e^y(\sin x - 1 - y)\right) =$$

$$= e^y(\sin x - 1 - y) - e^y =$$

$$= e^y(\sin x - 2 - y)$$

$$H_f(a) = \begin{bmatrix} -(1+e^y)\sin x & e^y \cos x \\ e^y \cos x & e^y(\sin x - 2 - y) \end{bmatrix}$$

V stacionarnih točkah velja $\cos x = 0$

1) $x = \frac{\pi}{2} + 2k\pi \Rightarrow \sin x = 1$

$$\boxed{H_f(a) = \begin{bmatrix} -\underbrace{(1+e^y)}_{>0} & 0 \\ 0 & -\underbrace{e^y}_{>0} \end{bmatrix}}$$

V stacionarnih točkah velja

i) $\frac{\partial f}{\partial x} = (1+e^y)\cos x = 0$

$\Rightarrow \cos x = 0$

ii) $\frac{\partial f}{\partial y} = e^y(\sin x - 1 - y) = 0$

$\Rightarrow \sin x - 1 - y = 0$

$\Rightarrow \frac{\partial^2 f}{\partial y^2} = e^y(\sin x - 2 - y) =$

$= e^y(\underbrace{\sin x - 1 - y}_{=0 \text{ v stacionarnih to.}}) - e^y$

$$\hookrightarrow H_f(a) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} ; \quad \lambda_1, \lambda_2 < 0$$

$$\det H_f(a) = \lambda_1 \lambda_2 > 0$$

$\Rightarrow$ $\underset{\substack{\shortparallel \\ \left(\frac{\pi}{2}+2k\pi,\ 0\right)}}{a}$ je maksimum

2) $x = -\frac{\pi}{2} + 2k\pi$; $\cos x = 0$, $\sin x = -1$

$\hookrightarrow H_f(a) = \begin{bmatrix} 1+e^y & 0 \\ 0 & -e^y \end{bmatrix}$

$\hookrightarrow \det H_f(a) < 0 \Rightarrow a$ ni ekstrem!

N2: Funkcija dveh spremenljivk je podana s predpisom

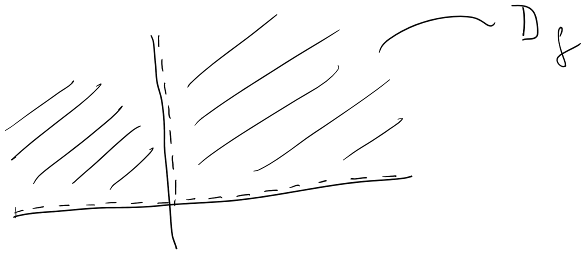
$$f(x,y) = \underline{x\ln(x^2)} + \underline{y^2\ln y}.$$

a) Poišči in skiciraj D_f.

b) Poišči in klasificiraj lokalne ekstreme.

R2: a) $\rightsquigarrow \begin{cases} x \in (-\infty,0)\cup(0,\infty) = \mathbb{R}\setminus\{0\} \\ y \in (0,\infty) \end{cases}$

$\hookrightarrow D_f = \mathbb{R}\setminus\{0\} \times (0,\infty)$



b) Poiščemo stacionarne točke:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x\ln x^2 + \underline{y^2\ln y}\right) =$$

i) $\underline{x>0} \rightsquigarrow x\ln x^2 = 2x\ln x$ $= \underline{\ln x^2 + 2}$

$$\frac{\partial f}{\partial x} = 2\cdot\ln x + 2\cancel{x}\frac{1}{\cancel{x}} = 2(\ln x + 1)$$

ii) $\underline{x<0} \rightsquigarrow x^2 = (-x)^2$; $x\ln x^2 =$
$= 2x\ln(-x)$

$$\frac{\partial f}{\partial x} = 2\ln(-x) + 2\cancel{x}\,\frac{1}{\cancel{-x}}\cdot(-1) =$$

$$= 2\ln(-x) + 2 = \underline{\ln x^2 + 2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(y^2 \ln y\right) = 2y\ln y + y^2\cdot\frac{1}{y} =$$

$$= 2y\ln y + y = y(\ln y^2 + 1)$$

- $$\frac{\partial f}{\partial x} = \boxed{2 + \ln x^2 = 0}$$

$$\ln x^2 = -2 \quad / e^{\cdot}$$

$$x^2 = e^{-2}$$

$$|x| = e^{-1} \rightsquigarrow x = \pm\frac{1}{e}$$

- $$\frac{\partial f}{\partial y} = y(\ln y^2 + 1) = 0 \rightarrow y > 0 / \ldots \text{ delimo z } y$$

1) $y = 0 \rightarrow \rightsquigarrow$ ni v D_f

2) $\ln y^2 + 1 = 0$

$$\ln y^2 = -1 \quad / e^{\cdot}$$

$$y^2 = e^{-1} \quad / \quad y > 0$$

$$y = e^{-1/2} = 1/\sqrt{e}$$

Stacionarni točki sta $\{(1/e, 1/\sqrt{e}), (-1/e, 1/\sqrt{e})\}$

Izračunamo še druge odvode:

- $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(2 + \ln x^2\right) = \frac{1}{x^2}\cdot 2x = \frac{2}{x}$$

- $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(2 + \ln x^2\right) = 0$$

- $$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(y + y\ln y^2\right) = 1 + 1\cdot\ln y^2 + \cancel{y}\cdot\frac{1}{y^{\cancel{2}}}\cdot 2y =$$

$$= 1 + \ln y^2 + 2 =$$

$$3 + \ln y^2 =$$

$$= 1 + \ln y^2 + 2 =$$
$$= 3 + \ln y^2 =$$
$$= 3 + 2\ln y$$

$\hookrightarrow \quad H_f(x,y) = \begin{bmatrix} 2/x & 0 \\ 0 & 3+2\ln y \end{bmatrix}$

$\hookrightarrow$ v stacionarnih tč.:

$x = \pm e^{-1} \rightsquigarrow 2/x = \pm 2e$

$y = e^{-1/2} \rightsquigarrow 3 + 2\ln y = 3 + 2\cdot(-1/2) =$
$= 3 - 1 = 2$

$$H_f(\pm e^{-1}, e^{-1/2}) = \begin{bmatrix} \pm 2e & 0 \\ 0 & 2 \end{bmatrix}$$

$\det H_f(a) = \pm 4e$

$\Rightarrow$ v točki $(-e^{-1}, e^{-1/2})$ <u>ni</u> ekstrema

v točki $(e^{-1}, e^{-1/2})$ je lokalni minimum

<u>Vezani ekstremi</u>

- $f, g : \mathbb{R}^n \to \mathbb{R}$; $M = \{x \in \mathbb{R}^n \mid g(x) = 0\}$;

Recimo, da za vse $x \in M$ velja, da je

$\nabla g(x) \neq 0$.

- Želimo poiskati ekstreme $f|_M$.

- Poiščemo točke $a_i \in M$ v katerih velja

- Poiščemo točke $a \in M$ v katerih velja

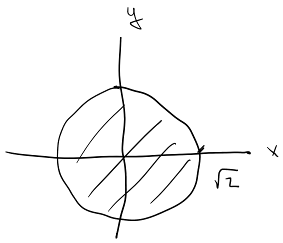
$$\frac{\partial f}{\partial x_i}(a) = \lambda \cdot \frac{\partial g}{\partial x_i}(a) \; ; \quad \lambda \in \mathbb{R} \setminus \{0\}$$

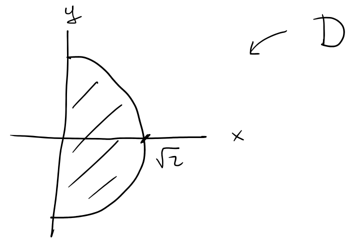
in zgornje velja za $i = 1, \dots, n$; $g(a) = 0$.

- Z izračunom vrednosti $f(a)$ preverimo, ali je v a ekstrem.

N3: Poišči globalne ekstreme funkcije $f(x,y) = x^2 - 2x + 4y$ na območju $D = \{(x,y) \mid x^2 + y^2 \le 2, \; x \ge 0\}$.

R3:





i) Najprej preverimo, ali ima f ekstreme (v običajnem smislu) na notranjosti D:

$$\mathring{D} = \{x^2 + y^2 < 2, \; x > 0\}.$$

↳ stacionarne tč.

$$\frac{\partial f}{\partial x} = 2x - 2 = 0 \iff x = 1$$

$$\frac{\partial f}{\partial y} = 8y = 0 \iff y = 0$$

↳ $a = (1,0) \in \mathring{D}$

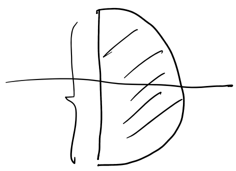
$$\frac{\partial^2 f}{\partial x^2} = 2 \ ; \ \frac{\partial^2 f}{\partial x \partial y} = 0 \ ; \ \frac{\partial^2 f}{\partial y^2} = 8$$

$$H_f(a) = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \rightsquigarrow \text{v } a \text{ imamo lokalni minimum;}$$

Velja $\underline{f(a)} = 1^2 - 2 \cdot 1 + 4 \cdot 0^2 =$

$$= 1 - 2 = \underline{\underline{-1}}$$

ii) Preverimo, ali ima f vezane ekstreme na ∂D



$\rightsquigarrow \partial D = D_1 \cup D_2$

$$D_1 = \{ (0, y) \mid -\sqrt{2} \le y \le \sqrt{2} \}$$

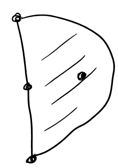
$$D_2 = \{ (x, y) \mid x \ge 0, \ \underline{x^2 + y^2 = 2} \}$$

1) Na D_1 lahko f zapišemo kot funkcijo ene spremenljivke:

$$(x, y) \in D_1 \Rightarrow f(x, y) = f(0, y) = \underset{\sim}{4y^2}$$

ekstremi: $f'(y) = 8y \ \cancel{} = 0$

$$\Rightarrow y = 0$$



$$f(0, 0) = 0$$

- $f(0, \pm\sqrt{2}) = 4 \cdot (\pm\sqrt{2})^2 = 4 \cdot 2 = \underline{8}$

Ker ima f minimum v $(1, 0)$;

$f(1, 0) = -1$, ter je

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$f(0, \pm\sqrt{2}) = 8$, je

$$f(1,0) < f(0,0) < f(0, \pm\sqrt{2})$$

$\Rightarrow$ v $(0,0)$ <u>ni</u> globalni ekstrem.

Vezani ekstremi na D_2 ?

$$D_2 = \{ (x,y) \mid g(x,y) = 0 , \ x \geq 0 \},$$

kjer je $g(x,y) = x^2 + y^2 - 2$

$$g(x,y) = 0 \iff x^2 + y^2 = 2$$

$$\frac{\partial g}{\partial x} = 2x \ ; \quad \frac{\partial g}{\partial y} = 2y$$

Iščemo (x,y) ; $x \geq 0$, da bo veljalo

$$2x - 2 = \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} = \lambda \cdot 2x$$

$$8y = \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} = \lambda \cdot 2y$$

$$g(x,y) = 0 .$$

$$2x - 2 = 2\lambda x$$

$$(2 - 2\lambda) x = 2$$

$$x = \frac{1}{1-\lambda}$$

$$8y = 2\lambda y$$

$$4y = \lambda y$$

$\rightsquigarrow$ $y = 0$ ali $\lambda = 4$

1) $y = 0$ $\rightsquigarrow$ $x^2 + \overset{0}{y}{}^2 = 2$, $x \geq 0$

$x = \sqrt{2}$

$\rightsquigarrow$ $f(\sqrt{2}, 0) = \sqrt{2}^2 - 2\sqrt{2} = 2 - 2\sqrt{2} > -1$

$\rightsquigarrow$ $(\sqrt{2}, 0)$ ni globalni ekstrem

$\Rightarrow$ $(\sqrt{2}, 0)$ ni globalni ekstrem

2) $y \neq 0$, $\lambda = 4$

$$x = \frac{1}{1-4} = -\frac{1}{3} \quad \dots \text{ ni v } D$$

Povzetek: Največjo vrednost doseže f v točkah $a_{1,2}(0, \pm\sqrt{2})$, $f(a_{1,2}) = 8$; najmanjšo vrednost pa doseže v $b = (1, 0)$; $f(b) = -1$.