Matematična analiza - 2. kolokvij

13. 6. 2023

Vsi odgovori morajo biti utemeljeni in pot do rešitve naj bo jasna. Vse naloge so enakovredne.

1. (a) Izračunajte
$$\int \frac{1}{(1-s)\sqrt{s}}\, ds.$$
(Namig: nova spremenljivka).

$$I = \int \frac{2\cdot 1}{2\cdot(1-s)\sqrt{s}}\, ds = \int \frac{2}{1-t^2}\, dt = \int \frac{2}{(1-t)(1+t)}\, dt = A\cdot \ln|1-t| + B\cdot \ln|1+t| + C$$

$$t = \sqrt{s} \Rightarrow s = t^2$$

$$dt = \frac{1}{2\sqrt{s}}\, ds$$

POSREDNI ODVOD !

$$\frac{A\cdot\boxed{(-1)}}{1-t} + \frac{B}{1+t} =$$

$$= \frac{-A(1+t) + B(1-t)}{1-t^2} = \frac{-A - At + B - Bt}{1-t^2}$$

$$-A - B = 0 \rightarrow -A = B$$

$$-A + B = 2 \rightarrow 2B = 2$$

$$B = 1,\ A = -1$$

$$\Rightarrow \quad I = -\ln|1-t| + \ln|1+t| + C$$

$$= \ln|1+\sqrt{s}| - \ln|1-\sqrt{s}| + C$$

$$= \ln\left|\frac{1+\sqrt{s}}{1-\sqrt{s}}\right| + C$$

(b) Izračunajte

$$\int \frac{x\cos x}{\sin^2 x}\,dx.$$

(Namig: na primernem mestu si pomagajte s točko a)).

$$\int \frac{x \cos x}{\sin^2 x}\,dx = \int \frac{\arcsin t}{t^2}\,dt$$

$t = \sin x \rightarrow x = \arcsin t$

$dt = \cos x\, dx$

PER PARTES

$$= -\frac{\arcsin t}{t} + \int \frac{1}{t\sqrt{1-t^2}}\,dt =$$

$u = \arcsin t \qquad du = \frac{1}{\sqrt{1-t^2}}\,dt$

$dv = \frac{1}{t^2}\,dt \qquad v = -\frac{1}{t}$

$$\left(\int \frac{1}{t^2}\,dt = \int t^{-2}\,dt = -t^{-1} + C\right)$$

$$-\frac{x}{\sin x}$$

$$= -\frac{\arcsin t}{t} + \int \frac{1 \cdot (-2t)}{t\sqrt{1-t^2}\cdot(-2t)}\,dt$$

ds

$$\left(\frac{-2t}{-2t^2\sqrt{1-t^2}}\,dt\right)$$

$$t^2 = 1 - s \leftarrow \quad s = 1 - t^2$$
$$ds = -2t\,dt$$

$$= -\frac{x}{\sin x} - \frac{1}{2}\int \frac{1}{(1-s)\cdot\sqrt{s}}\,ds$$

a)
$$= -\frac{x}{\sin x} - \frac{1}{2}\cdot \ln\left|\frac{1+\sqrt{s}}{1-\sqrt{s}}\right| + C$$

$$s = 1 - t^2 = 1 - \sin^2 x = \cos^2 x$$

$$= -\frac{x}{\sin x} - \frac{1}{2}\cdot \ln\left|\frac{1+\cos x}{1-\cos x}\right| + C$$

ŠE ENA POT:

PER PARTES

$$\int \frac{x\cos x}{\sin^2 x}\,dx = -\frac{x}{\sin x} - \int -\frac{1}{\sin x}\,dx$$

$$u = x \qquad du = dx$$

$$dv = \frac{\cos x}{\sin^2 x}\,dx \qquad v = -\frac{1}{\sin x}$$

$$\int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{t^2}\,dt = -\frac{1}{t} + C$$
$$= -\frac{1}{\sin x} + C$$

$$t = \sin x$$
$$dt = \cos x\,dx$$

$$= -\frac{x}{\sin x} - \int \frac{\boxed{-\sin x}}{\sin^2 x}\,\boxed{dx} = -\frac{x}{\sin x} - \int \frac{1}{1-t^2}\,dt =$$

$$t = \cos x$$
$$dt = -\sin x\,dx$$

$$\sin^2 x = 1-\cos^2 x = 1-t^2$$

PODOBNO KOT PRI a) TOČKI

$$= -\frac{x}{\sin x} - \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + C$$

$$= -\frac{x}{\sin x} - \frac{1}{2}\ln\left|\frac{1+\cos x}{1-\cos x}\right| + C$$

2. Podana je funkcija $f(x) = \frac{e^x}{e^{2x}+1}$.

(a) Izračunajte ploščino med absciso in grafom funkcije nad intervalom $[0, \infty)$.

(b) Izračunajte volumen vrtenine, ki jo dobimo, če graf funkcije f nad intervalom $[0, \infty)$ zavrtimo okrog abscise.

Opazimo: $f(x) > 0 \quad \forall x$

a) $S = \int_0^\infty \frac{e^x}{e^{2x}+1}\,dx = \int_1^\infty \frac{1}{t^2+1}\,dt = \arctan t \Big|_1^\infty = \text{"}\arctan \infty\text{"} - \arctan 1$

$= \lim_{t\to\infty} \arctan t = \frac{\pi}{2}$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

$t = e^x$

$dt = e^x\,dx$

MEJE (!) $x = 0 \to t = 1$

"$x = \infty \to t = \infty$"

b) $V = \pi \cdot \int_0^\infty \left(\frac{e^x}{e^{2x}+1}\right)^2 dx = \frac{\pi}{2} \int_0^\infty \frac{2 \cdot e^{2x}}{(e^{2x}+1)^2}\,dx = \frac{\pi}{2} \cdot \int_2^\infty \frac{1}{t^2}\,dt = \frac{\pi}{2}\left(-\frac{1}{t}\Big|_2^\infty\right)$

$t = e^{2x} + 1$

$dt = 2e^{2x}\,dx$

MEJE (!) $x = 0 \to t = 2$

"$x = \infty \to t = \infty$"

$= \frac{\pi}{2} \cdot \left(0 + \frac{1}{2}\right) = \frac{\pi}{4}$

$\lim_{t\to\infty} \left(-\frac{1}{t}\right) = 0$

3. Obravnavajte konvergenco posplošenega integrala

$$\int_1^\infty \frac{\ln x}{x^2-1}dx$$

SPOMNIMO SE:

• POL $x=a$: $f(x)=\frac{\varphi(x)}{(x-a)^s}$,

1. $s<1$, $\lim_{x\to a} \varphi(x) \in \mathbb{R} \Rightarrow \int_a^{\dots} f(x)dx$ obstaja pri polu

2. $s\geq 1$, $\lim_{x\to a} \varphi(x) \neq 0 \Rightarrow \int_a^{\dots} f(x)dx$ NE obstaja (pri polu)

• ∞: $f(x)=\frac{\varphi(x)}{x^s}$,

1. $s>1$, $\lim_{x\to\infty} \varphi(x) \in \mathbb{R} \Rightarrow \int_{\dots}^{\infty} f(x)dx$ obstaja v neskončnosti

2. $s\leq 1$, $\lim_{x\to\infty} \varphi(x) \neq 0 \Rightarrow \int_{\dots}^{\infty} f(x)dx$ NE obstaja (v nesk.)

Imamo 2 težavi:

- POL $x=1$: $f(x)=\dfrac{\ln x}{(x-1)(x+1)}$

Poskusimo očitno izbiro $f(x)=\dfrac{\overbrace{\dfrac{\ln x}{(x+1)}}^{\varphi(x)}}{(x-1)^1}$ $s=1$

$$\lim_{x\to1}\varphi(x)=\lim_{x\to1}\frac{\overset{\to 0}{\ln x}}{\underset{\to 2}{x+1}}=0$$ → NI DOBRA IZBIRA, NAM NE POVE NIČ

(Če bi vzeli $s>1$: $f(x)=\dfrac{\dfrac{\ln x}{(x+1)(x-1)}\cdot(x-1)^s}{(x-1)^s}=\dfrac{\overbrace{\dfrac{\ln x}{(x+1)}\cdot(x-1)^{s-1}}^{\varphi(x)}}{(x-1)^s}$

$$\lim_{x\to1}\varphi(x)=\lim_{x\to1}\frac{\overset{\to0}{\ln x}\cdot\overset{\to0}{(x-1)^{s-1}}}{\underset{\to2}{x+1}}=0$$ → -||-)

Torej se zdi, da bi moral obstajati - poskusimo s $s=\frac{1}{2}$:

$$f(x)=\frac{\overbrace{\dfrac{\ln x}{(x+1)\sqrt{x+1}}}^{\varphi(x)}}{\sqrt{x-1}}$$

$s=\frac{1}{2}$

$$\lim_{x\to 1} \frac{\overset{\to 0}{\ln x}}{\underset{\to 0}{(x+1)\sqrt{x-1}}} \overset{L'H}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{\sqrt{x-1} + (x+1)\cdot\frac{1}{2\sqrt{x-1}}} \frac{\cdot 2\sqrt{x-1}}{\cdot 2\sqrt{x-1}} =$$

$$= \lim_{x\to 1} \frac{\overset{\to 0}{\frac{2\sqrt{x-1}}{x}}}{\underset{\to 1}{2(x-1)+x+1}} = 0 \in \mathbb{R}$$

$\Rightarrow$ INTEGRAL OBSTAJA PRI POLU $x=1$

• ∞ :

$$f(x) = \frac{\overbrace{\frac{\ln x}{x^2-1}\cdot x^s}^{\varphi(x)}}{x^s}$$

(Opazujemo $\varphi(x) = \underbrace{\ln x}_{\to\infty} \cdot \underbrace{\frac{x^s}{x^2-1}}_{\to 0 \text{ ALI } 1 \text{ ALI } \infty}$.

$x\to\infty$:

$s<2$, $s=2$, $s>2$

Če $s\geq 2$, bo šel φ proti ∞. Poskusimo nek $s<2$, npr. $s=\frac{3}{2}$.)

$$\lim_{x\to\infty} \frac{\overbrace{\ln x \cdot x^{\frac{3}{2}}}^{\to\infty}}{\underbrace{x^2-1}_{\to\infty}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}\cdot x^{\frac{3}{2}} + \ln x \cdot \frac{3}{2}\, x^{\frac{1}{2}}}{2x} =$$

$$= \lim_{x\to\infty} \frac{\sqrt{x} + \frac{3}{2}\cdot\sqrt{x}\cdot \ln x}{2x} =$$

$$= \lim_{x\to\infty} \frac{\sqrt{x}\left(1+\frac{3}{2}\ln x\right)}{2x} = \lim_{x\to\infty} \frac{\overbrace{1+\frac{3}{2}\ln x}^{\to\infty}}{\underbrace{2\sqrt{x}}_{\to\infty}}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{3}{2x}}{\frac{\cancel{x}^{1}}{\cancel{x}\sqrt{x}}} = \lim_{x\to\infty} \frac{3}{2\sqrt{x}}$$

$$= 0 \in \mathbb{R}$$

$$s = \underset{1}{\overset{\frac{3}{2}}{\vee}}, \quad \lim_{x\to\infty} \varphi(x) \in \mathbb{R}$$

$\Rightarrow$ INTEGRAL OBSTAJA PRI ∞

$\Rightarrow$ Integral obstaja.

4. Funkcijo

$$f(x) = \frac{xe^{2x^3}}{2}$$

razvijte v Taylorjevo vrsto okoli točke 0. Zapišite predpis za koeficiente a_m, ki pripadajo potencam x^m, $m \geq 0$, v Taylorjevem razvoju. Določite vrednosti $f^{(2022)}(0)$ in $f^{(2023)}(0)$.

$$\circledast \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$f(x) = \frac{x}{2} \cdot e^{2x^3} \overset{\circledast}{=} \frac{x}{2} \cdot \sum_{n=0}^{\infty} \frac{(2x^3)^n}{n!} = \frac{x}{2} \cdot \sum_{n=0}^{\infty} \frac{2^n \cdot x^{3n}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x}{2} \cdot \frac{2^n \cdot x^{3n}}{n!} = \sum_{n=0}^{\infty} \frac{2^{n-1} \cdot x^{3n+1}}{n!}$$

$$m = 3n+1 \rightarrow n = \frac{m-1}{3}$$

$$n - 1 = \frac{m-4}{3}$$

NI x-ov!

$$f(x) = \sum_{m=0}^{\infty} a_m x^m$$

$$a_m = \begin{cases} \dfrac{2^{\frac{m-4}{3}}}{\left(\frac{m-1}{3}\right)!} & ; m = 3n+1 \\ 0 & ; \text{sicer} \end{cases}$$

$$f^{(m)}(0) = a_m \cdot m!$$

$$3 \nmid 2022$$

$$f^{(2022)}(0) = 0 \cdot 2022! = 0$$

$$f^{(2023)}(0) = \frac{2^{263}}{674!} \cdot 2023!$$

$$2023 = 3 \cdot 674 + 1$$