Matematična analiza - 1. izpit

13. 6. 2023

Vsi odgovori morajo biti utemeljeni in pot do rešitve naj bo jasna. Vse naloge so enakovredne.

1. Dana je funkcija $f(x) = \frac{x+2}{\sqrt{x^2+2}}$.

   (a) Poiščite ničle in pole funkcije f, zapišite njeno definicijsko območje in raziščite obnašanje funkcije na robovih območja.

   (b) Poiščite intervale naraščanja in padanja ter stacionarne točke funkcije f. Stacionarne točke tudi klasificirajte.

   (c) Poiščite intervale konveksnosti in konkavnosti ter prevoje funkcije f.

   (d) Čimbolj natančno skicirajte graf funkcije f s pomočjo prejšnjih točk.

a) NIČLE: $\frac{x+2}{\sqrt{x^2+2}} = 0$

$x+2=0$

$\underline{x=-2}$

POLI: $\sqrt{x^2+2} = 0$

$x^2+2=0$

$x^2=-2$

NI REŠITVE (gledamo le $x \in \mathbb{R}$)

Ker je $x^2+2>0 \;\; \forall x$, je $\sqrt{x^2+2}$ definiran in $\neq 0$ pri vseh x $\rightarrow D_f = \mathbb{R}$

$$\lim_{x\to\pm\infty} \frac{x+2}{\sqrt{x^2+2}} = \lim_{x\to\pm\infty} \frac{x\left(1+\frac{2}{x}\right)}{\sqrt{x^2}\cdot\sqrt{1+\frac{2}{x^2}}}$$

$\sqrt{x^2} = |x|$

$$= \lim_{x \to \pm\infty} \frac{x}{|x|} \cdot \frac{\left(1+\frac{2}{x}\right)}{\sqrt{1+\frac{2}{x^2}}}$$

$x>0$ // $\quad$ // $x<0$ $\quad$ ↓

$1 \quad -1 \quad 1$

$$\Rightarrow \lim_{x\to\infty} f(x) = 1$$

$$\lim_{x\to-\infty} f(x) = -1$$

b) $$f'(x) = \frac{\sqrt{x^2+2} - (x+2)\frac{1\cdot \cancel{2}x}{\cancel{2}\sqrt{x^2+2}}}{x^2+2} \cdot \frac{\sqrt{x^2+2}}{\sqrt{x^2+2}} = \frac{x^2+2-\overbrace{(x+2)\cdot x}^{-(x^2+2x)}}{(x^2+2)^{\frac{3}{2}}}$$

$$= \frac{2-2x}{(x^2+2)^{\frac{3}{2}}}$$

$$\frac{2-2x}{(x^2+2)^{\frac{3}{2}}} > 0 \quad \Big/ \cdot (x^2+2)^{\frac{3}{2}} > 0$$

$$2-2x > 0$$

$$\underline{x < 1}$$

rastuće: $x \in (-\infty, 1)$

padanje: $x \in (1, \infty)$

stac. točke: $x = 1$, ↗ ↘ LOK. MAX

c) $f''(x) = \left(\dfrac{2-2x}{(x^2+2)^{\frac{3}{2}}} \right)' = \dfrac{-2\cdot(x^2+2)^{\frac{3}{2}} - (2-2x)\cdot\frac{3}{\cancel{2}}\cdot(x^2+2)^{\frac{1}{2}}\cdot\cancel{2}x}{(x^2+2)^3} \quad : \sqrt{x^2+2}$

$\quad\quad : \sqrt{(x^2+2)}$

$$= \frac{-2\cdot(x^2+2) - (6x-6x^2)}{(x^2+2)^{\frac{5}{2}}}$$

$$= \frac{-2x^2-4-6x+6x^2}{(x^2+2)^{\frac{5}{2}}}$$

$$= \frac{4x^2-6x-4}{(x^2+2)^{\frac{5}{2}}}$$

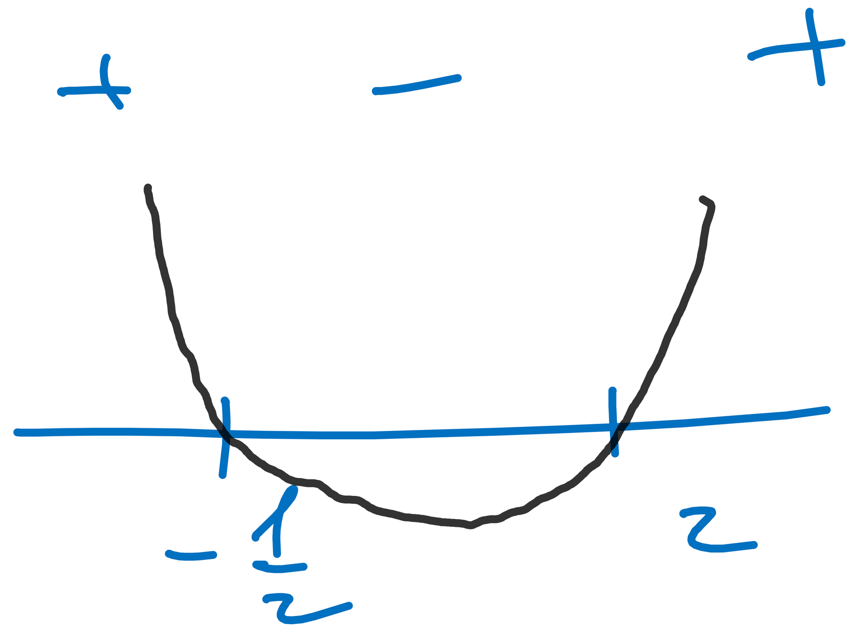
$$\frac{4x^2-6x-4}{(x^2+2)^{\frac{5}{2}}} > 0 \quad \Big| \cdot (x^2+2)^{\frac{5}{2}} > 0$$

$4x^2-6x-4 > 0 \quad \Big| : 2$

$\underline{2}x^2-3x-2 > 0$

$2 > 0 \Rightarrow$ OBLIKA $\cup$

$x_{1,2} = \dfrac{3 \pm 5}{4}$ $\quad x_1 = 2$, $\quad x_2 = -\frac{1}{2}$



konveksna:

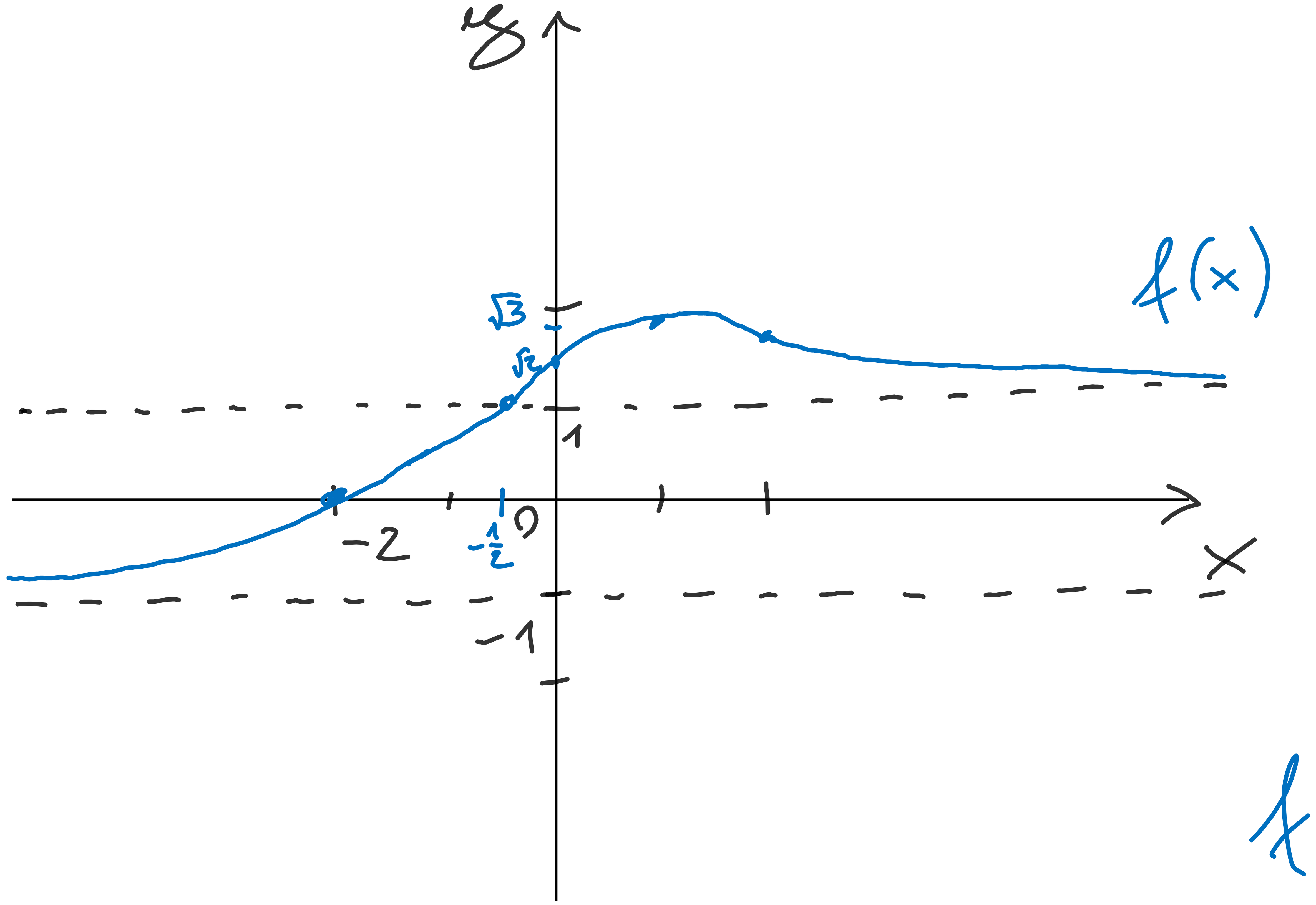
$x \in (-\infty, -\frac{1}{2}) \cup (2, \infty)$

konkavna:

$x \in (-\frac{1}{2}, 2)$

prevoji: $x = -\frac{1}{2}$, $x = 2$

d)



$f(0) = \frac{2}{\sqrt{2}} = \sqrt{2}$

$f(1) = \frac{3}{\sqrt{3}} = \sqrt{3}$

$f(2) = \frac{4}{\sqrt{6}} \doteq 1{,}5$

$f(x) = 1$

$\frac{x+2}{\sqrt{x^2+2}} = 1$

$x + 2 = \sqrt{x^2+2} \quad |^2 \quad (x > -2)$

$\cancel{x^2} + 4x + 4 = \cancel{x^2} + 2$

$4x = -2$

$\boxed{x = -\frac{1}{2}}$ ✓

$f(x) = -1$

$x + 2 = -\sqrt{x^2+2}$

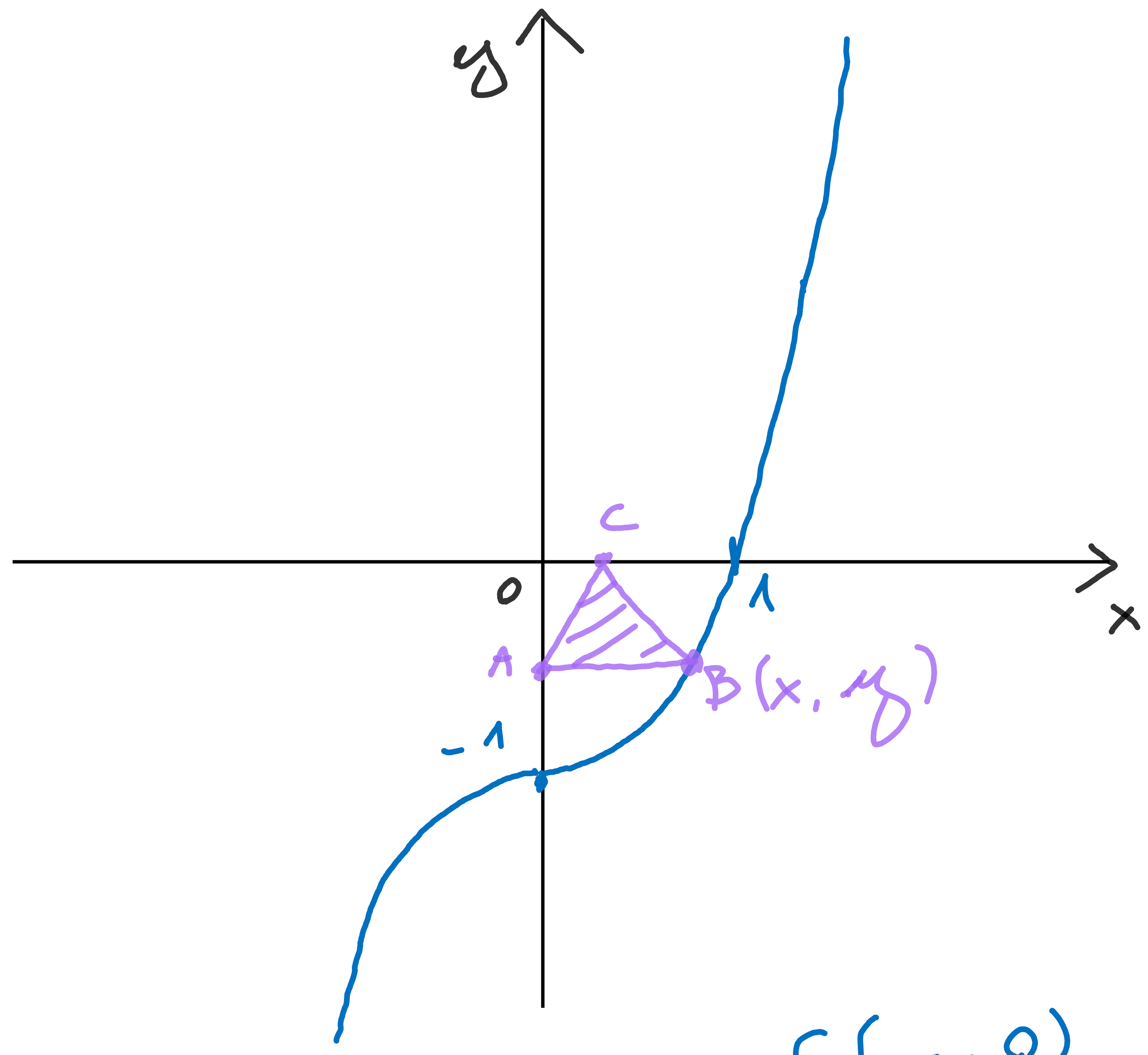
$-(x+2) = \sqrt{x^2+2} \quad |^2 \quad (x < -2)$

...

$x = -\frac{1}{2}$ // ní řešení

2. V omejeno območje, ki ga omejujejo krivulja $y - x^3 + 1 = 0$ ter koordinatni osi, želimo včrtati trikotnik tako, da bo imel osnovnico vzporedno z x-osjo, vrh pa bo ležal na x-osi. Med takimi trikotniki poiščite tistega z največjo ploščino, ter to ploščino izračunajte.

$y = x^3 - 1$



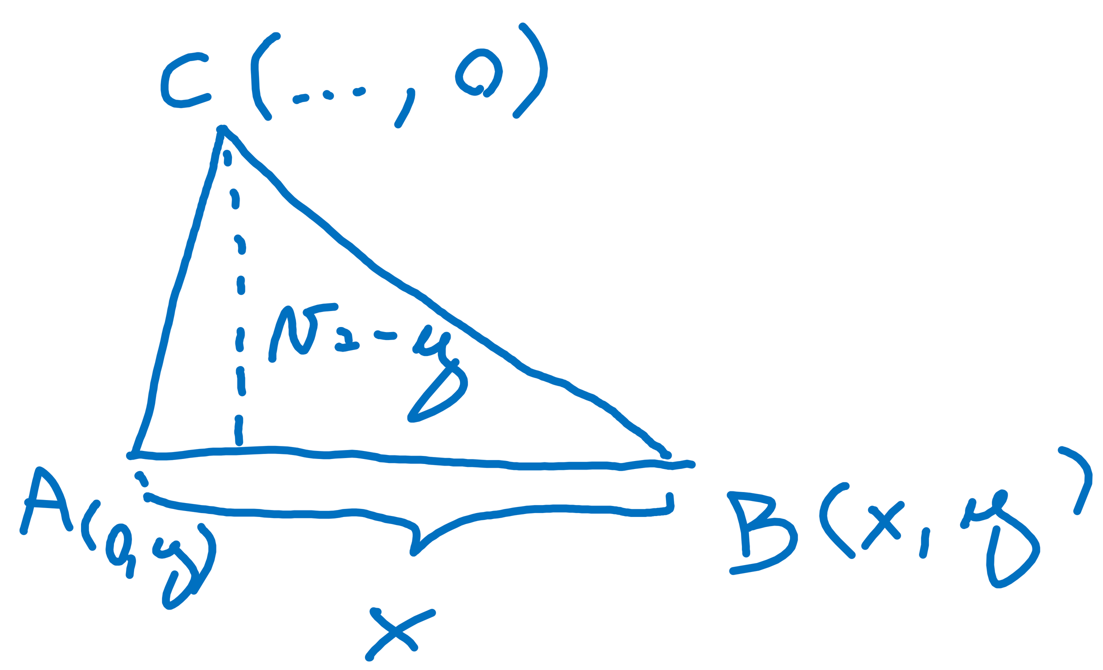
na krivulji

$B(x, y) \rightarrow B(x, x^3 - 1)$

$A(0, y)$

$|AB| = x$

$y < 0$, višina mora biti > 0



$$S = \frac{|AB| \cdot v}{2} = \frac{x \cdot (-y)}{2}$$

$$S(x) = \frac{x \cdot (1 - x^3)}{2} = \frac{x - x^4}{2}$$

$$S'(x) = \frac{1}{2}\left(1 - 4x^3\right) = 0$$

$$1 - 4x^3 = 0$$

$$x^3 = \frac{1}{4}$$

$$x = \sqrt[3]{\frac{1}{4}}$$

$$S\left(\sqrt[3]{\frac{1}{4}}\right) = \frac{\sqrt[3]{\frac{1}{4}} \cdot \left(1 - \frac{1}{4}\right)}{2} = \frac{3}{8} \cdot \sqrt[3]{\frac{1}{4}}$$

3. (a) Izračunajte

$$\int \frac{1}{(1-s)\sqrt{s}}\,ds.$$

(Namig: nova spremenljivka).

(b) Izračunajte

$$\int \frac{x\cos x}{\sin^2 x}\,dx.$$

(Namig: na primernem mestu si pomagajte s točko a)).

a)

$$I = \int \frac{2\cdot 1}{2\cdot(1-s)\sqrt{s}}\,ds = \int \frac{2}{1-t^2}\,dt = \int \frac{2}{(1-t)(1+t)}\,dt = A\cdot \ln|1-t| + B\cdot \ln|1+t| + C$$

$$t = \sqrt{s} \Rightarrow s = t^2$$

$$dt = \frac{1}{2\sqrt{s}}\,ds$$

POSREDNI ODVOD !

$$\frac{A\cdot\boxed{(-1)}}{1-t} + \frac{B}{1+t} = \frac{-A(1+t) + B(1-t)}{1-t^2} = \frac{-A - At + B - Bt}{1-t^2}$$

$$-A - B = 0 \rightarrow -A = B$$

$$-A + B = 2 \rightarrow 2B = 2$$

$$B = 1,\ A = -1$$

$$\Rightarrow \quad I = -\ln|1-t| + \ln|1+t| + C$$

$$= \ln|1+\sqrt{s}| - \ln|1-\sqrt{s}| + C$$

$$= \ln\left|\frac{1+\sqrt{s}}{1-\sqrt{s}}\right| + C$$

b)

$$\int \frac{x\cos x}{\sin^2 x}\,dx = \int \frac{\arcsin t}{t^2}\,dt \overset{\text{PER PARTES}}{=} -\frac{\arcsin t}{t} + \int \frac{1}{t\sqrt{1-t^2}}\,dt =$$

$t = \sin x \rightarrow x = \arcsin t$

$dt = \cos x\,dx$

$u = \arcsin t \qquad du = \frac{1}{\sqrt{1-t^2}}\,dt$

$dv = \frac{1}{t^2}\,dt \qquad v = -\frac{1}{t}$

$$\left(\int \frac{1}{t^2}\,dt = \int t^{-2}\,dt = -t^{-1} + C\right)$$

$$-\frac{x}{\sin x}$$

$$= -\frac{\arcsin t}{t} + \int \frac{1 \cdot (-2t)}{t\sqrt{1-t^2}\cdot(-2t)}\,dt$$

ds

$$\left(\frac{-2t}{-2t^2\sqrt{1-t^2}}\,dt\right)$$

$$t^2 = 1 - s \leftarrow \quad s = 1 - t^2$$

$$ds = -2t\,dt$$

$$= -\frac{x}{\sin x} - \frac{1}{2}\int \frac{1}{(1-s)\cdot\sqrt{s}}\,ds$$

$$\overset{a)}{=} -\frac{x}{\sin x} - \frac{1}{2}\cdot \ln\left|\frac{1+\sqrt{s}}{1-\sqrt{s}}\right| + C$$

$$s = 1 - t^2 = 1 - \sin^2 x = \cos^2 x$$

$$= -\frac{x}{\sin x} - \frac{1}{2}\cdot \ln\left|\frac{1+\cos x}{1-\cos x}\right| + C$$

ŠE ENA POT:

$$\int \frac{x\cos x}{\sin^2 x}\,dx \overset{\text{PER PARTES}}{=} -\frac{x}{\sin x} - \int -\frac{1}{\sin x}\,dx$$

$$u = x \qquad du = dx$$

$$dv = \frac{\cos x}{\sin^2 x}\,dx \qquad v = -\frac{1}{\sin x}$$

$$\int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{t^2}\,dt = -\frac{1}{t} + C$$

$$= -\frac{1}{\sin x} + C$$

$$t = \sin x$$

$$dt = \cos x\,dx$$

$$= -\frac{x}{\sin x} - \int \frac{-\sin x}{\sin^2 x}\,dx = -\frac{x}{\sin x} - \int \frac{1}{1-t^2}\,dt =$$

$$t = \cos x$$

$$dt = -\sin x\,dx$$

$$\sin^2 x = 1-\cos^2 x = 1-t^2$$

PODOBNO KOT PRI a) TOČKI

$$= -\frac{x}{\sin x} - \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + C$$

$$= -\frac{x}{\sin x} - \frac{1}{2}\ln\left|\frac{1+\cos x}{1-\cos x}\right| + C$$

4. Funkcijo

$$f(x) = \frac{xe^{2x^3}}{2}$$

razvijte v Taylorjevo vrsto okoli točke 0. Zapišite predpis za koeficiente a_m, ki pripadajo potencam x^m, $m \geq 0$, v Taylorjevem razvoju. Določite vrednosti $f^{(2022)}(0)$ in $f^{(2023)}(0)$.

$(*)$ $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$f(x) = \frac{x}{2} \cdot e^{2x^3} \overset{(*)}{=} \frac{x}{2} \cdot \sum_{n=0}^{\infty} \frac{(2x^3)^n}{n!} = \frac{x}{2} \cdot \sum_{n=0}^{\infty} \frac{2^n \cdot x^{3n}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x}{2} \cdot \frac{2^n \cdot x^{3n}}{n!} = \sum_{n=0}^{\infty} \frac{2^{n-1} \cdot x^{3n+1}}{n!}$$

$m = 3n+1 \rightarrow n = \frac{m-1}{3}$

$n - 1 = \frac{m-4}{3}$

NI x-ov!

$$f(x) = \sum_{m=0}^{\infty} a_m x^m$$

$$a_m = \begin{cases} \dfrac{2^{\frac{m-4}{3}}}{\left(\frac{m-1}{3}\right)!} & ; m = 3n+1 \\ 0 & ; \text{sicer} \end{cases}$$

$3 \nmid 2022$

$$f^{(m)}(0) = a_m \cdot m!$$

$$f^{(2022)}(0) = 0 \cdot 2022! = 0$$

$$f^{(2023)}(0) = \frac{2^{263}}{674!} \cdot 2023!$$

$2023 = 3 \cdot 674 + 1$