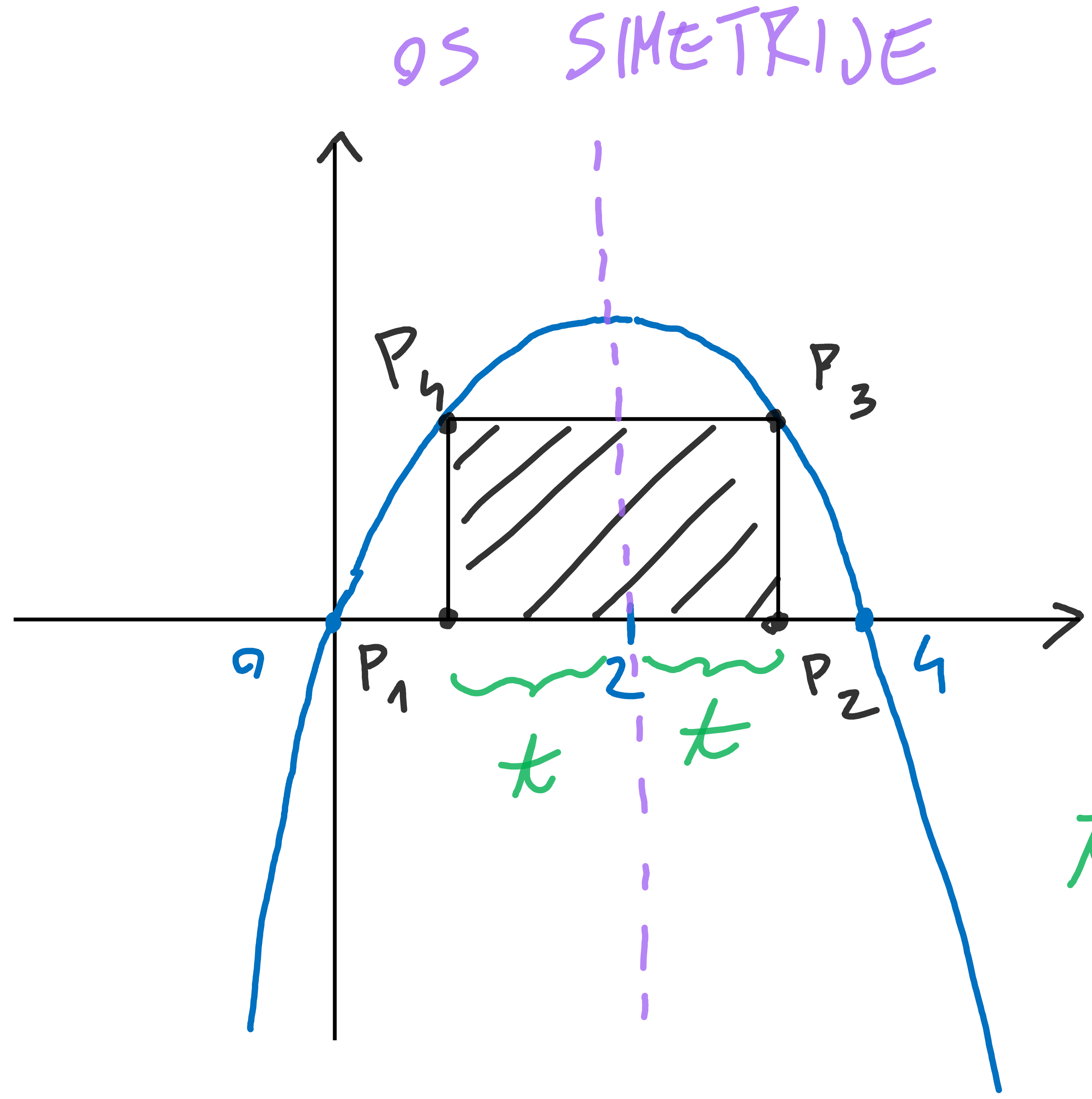
1. V območje omejeno s krivuljo $y = 8x - 2x^2$ ter ordinatno osjo včrtamo pravokotnik tako, da ena stranica leži na abscisni osi. Poiščite tistega izmed takih pravokotnikov, ki ima največjo ploščino. Zapišite koordinate vseh štirih oglišč ter ploščino tega pravokotnika.

(Namig: pomagajte si s simetrijo dane krivulje.)



$t > 0$

$$y = 8x - 2x^2 = 2x(4-x)$$

NIČLE: $x_1 = 0$, $x_2 = 4$

$P_1(x_L, 0)$, $P_2(x_D, 0)$

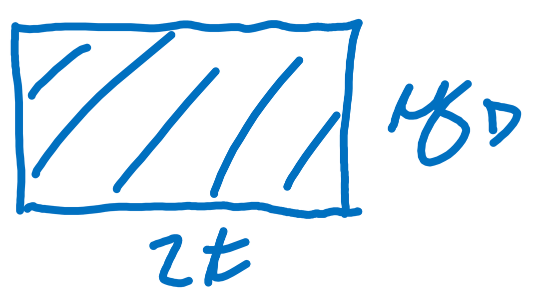
$P_4(x_L, y_L)$, $P_3(x_D, y_D)$

$$y_D = 2x_D(4 - x_D) \qquad x_D = 2 + t$$

$$= 2\cdot(2+t)(4-(2+t))$$

$$= 2\cdot(2+t)(2-t) = 2\cdot(4-t^2)$$

STRANICA VZP. X-OSI: $y_L = y_D$



$$S(t, y_D) = 2t \cdot y_D$$

$$S(t) = 2t\cdot 2(4-t^2) = 16t - 4t^3$$

$$(= 4t(4-t^2))$$

$$S'(t) = 16 - 12t^2 = 0$$

$$t^2 = \frac{16}{12} = \frac{4}{3}$$

$$\boxed{t = \pm \frac{2}{\sqrt{3}}}$$

$$S\left(\frac{2}{\sqrt{3}}\right) = 4 \cdot \frac{2}{\sqrt{3}} \cdot \left(4 - \frac{4}{3}\right) = \frac{8\sqrt{3}}{3} \cdot \frac{8}{3} = \underline{\underline{\frac{64\sqrt{3}}{9}}}$$

$$x_C = 2 - t = 2 - \frac{2}{\sqrt{3}} = 2 - \frac{2\sqrt{3}}{3} = \frac{6-2\sqrt{3}}{3} \qquad y_D = 2 \cdot \left(4 - \frac{4}{3}\right) = \frac{16}{3}$$

$$x_D = 2 + t = \frac{6+2\sqrt{3}}{3}$$

$$P_1\left(\frac{6-2\sqrt{3}}{3}, 0\right), \quad P_2\left(\frac{6+2\sqrt{3}}{3}, 0\right), \quad P_3\left(\frac{6+2\sqrt{3}}{3}, \frac{16}{3}\right), \quad P_4\left(\frac{6-2\sqrt{3}}{3}, \frac{16}{3}\right)$$

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2. Podana je funkcija s predpisom

$$f(x) = \begin{cases} 2\arctan(x) + a; & x < 0, \\ bx + c; & 0 \leq x < 1, \\ x^2; & x \geq 1. \end{cases}$$

Določite vrednosti parametrov $a, b, c \in \mathbb{R}$, da bo funkcija zvezno odvedljiva. Pri dobljenih parametrih tudi zapišite odvod funkcije.

(Namig: pri $x = 0$ in $x = 1$ izračunajte levi in desni odvod po definiciji.)

• zveznost v $x=0$:

$$\left.\begin{aligned} \lim_{x \uparrow 0} f(x) &= \lim_{x \uparrow 0} 2\arctan \overset{\nearrow 0}{x} + a = a \\ \lim_{x \downarrow 0} f(x) &= \lim_{x \downarrow 0} bx + c = c = f(0) \end{aligned}\right\} \Rightarrow \boxed{a = c} \; (1)$$

v $x=1$:

$$\left.\begin{aligned} \lim_{x \uparrow 1} f(x) &= \lim_{x \uparrow 1} bx + c = b + c \\ \lim_{x \downarrow 1} f(x) &= \lim_{x \downarrow 1} x^2 = 1 = f(1) \end{aligned}\right\} \Rightarrow \boxed{b + c = 1} \; (2)$$

• odvedljivost v $x=0$ (PO DEFINICIJI)

• Z LEVE: $\lim\limits_{h\uparrow 0} \frac{f(h)-f(0)}{h} = \lim\limits_{h\uparrow 0} \frac{2\cdot \arctan h + \overbrace{a-c}^{=0 \text{ ZARADI } (1)}}{h}$

$(f(0)=c)$

$\overset{L'H}{=} \lim\limits_{h\uparrow 0} \frac{2\cdot\frac{1}{1+h^2}}{1} = 2$

• Z DESNE: $\lim\limits_{h\downarrow 0} \frac{f(h)-f(0)}{h} = \lim\limits_{h\downarrow 0} \frac{b\cancel{h}+\cancel{c}-\cancel{c}}{\cancel{h}} = b$

$\boxed{b=2}$ (3)

Enačbe (1), (2) in (3) so dovolj, da določimo a, b, c:

$b=2$, $c=1-b=-1$, $a=c=-1$,

moramo pa preveriti še, če dobljeni a, b, c delujejo za odvedljivost v $x=1$:

• Z LEVE: $\lim\limits_{h\uparrow 0} \frac{f(1+h)-f(1)}{h} = \lim\limits_{h\uparrow 0} \frac{b\cdot(1+h)+c-1}{h} = \lim\limits_{h\uparrow 0} \frac{2\cdot(1+h)-1-1}{h} = \lim\limits_{h\uparrow 0} \frac{2\cancel{h}}{\cancel{h}} = 2$

$(f(1)=1)$

• Z DESNE: $\lim\limits_{h\downarrow 0} \frac{f(1+h)-f(1)}{h} = \lim\limits_{h\downarrow 0} \frac{(1+h)^2-1}{h} = \lim\limits_{h\downarrow 0} \frac{\cancel{1}+2h+h^2-\cancel{1}}{h} = \lim\limits_{h\downarrow 0} 2+h = 2$

$= \checkmark$

$$\Rightarrow f(x) = \begin{cases} 2\arctan x - 1 & ; x < 0 \\ 2x - 1 & ; 0 \le x < 1 \\ x^2 & ; x \ge 1 \end{cases}$$

je zvezno odvedljiva,
njen odvod je

$$f'(x) = \begin{cases} \dfrac{2}{1+x^2} & ; x < 0 \\ 2 & ; 0 \le x < 1 \\ 2x & ; x \ge 1 \end{cases}$$

$f'(0) = 2$ IN $f'(1) = 2$
SMO IZRAČUNALI PO DEF. IN
JIH LAHKO DODAMO V PREDPIS

3. (a) Izračunajte nedoločeni integral

$$\int \frac{9x}{x^3 - 3x + 2}\, dx.$$

(b) Izračunajte določeni integral

$$\int_0^3 \arctan \sqrt{x}\, dx.$$

a) $x^3 - 3x + 2$ $\rightarrow$

	1	0	-3	2
1		1	1	-2
	1	1	-2	0
1		1	2	
	1	2	0	

} VSOTA = 0 → NIČLA $x=1$

$x_{1,2} = 1$

$x + 2 = 0$

$x_3 = -2$

$\Rightarrow x^3 - 3x + 2 = (x-1)^2 (x+2)$

$$\int \frac{9x}{x^3 - 3x + 2}\, dx = \int \frac{9x}{(x-1)^2 (x+2)}\, dx$$

$$= \frac{A}{x-1} + B \cdot \ln|x-1| + C \cdot \ln|x+2| + D$$

t

$\downarrow '$

$$\frac{-A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$= \frac{-A(x+2) + B(x-1)(x+2) + C(x-1)^2}{(x-1)^2 (x+2)} = \frac{-Ax - 2A + Bx^2 + Bx - 2B + Cx^2 - 2Cx + C}{(x-1)^2 \cdot (x+2)}$$

$x^2: \quad B + C = 0 \quad \rightarrow \quad C = -B$

$x: \quad -A + B - 2C = 9$

$1: \quad -2A - 2B + C = 0$

$$\left.\begin{aligned} -A + 3B &= 9 \quad (*) \\ -2A - 3B &= 0 \end{aligned}\right\} +$$

$-3A = 9$

$\boxed{A = -3}$

$(*)$ $\boxed{B = \frac{9+A}{3} = 2}$ $\boxed{C = -2}$

$$\int \frac{9x}{(x-1)^2(x+2)}\,dx = \frac{-3}{x-1} + 2\ln|x-1| - 2\ln|x+2| + D$$

b) $\int_0^3 \arctan\sqrt{x}\,dx = x \cdot \arctan\sqrt{x}\Big|_0^3 - \int_0^3 \frac{x}{1+x} \cdot \frac{1}{2\sqrt{x}}\,dx = (*)$

$u = \arctan\sqrt{x} \quad du = \frac{1}{1+(\sqrt{x})^2} \cdot \underbrace{\frac{1}{2\sqrt{x}}}_{(\sqrt{x})'}\,dx$

$dv = dx \quad v = x$

$t = \sqrt{x}$

$dt = \frac{1}{2\sqrt{x}}\,dx$

$$\circledast = 3 \arctan \sqrt{3} - 0 - \int_0^{\sqrt{3}} \frac{1+t^2-1}{1+t^2}\,dt$$

$$= 3 \cdot \frac{\pi}{3} - \int_0^{\sqrt{3}} \left(1 - \frac{1}{1+t^2}\right) dt = \pi - \left(t - \arctan t \Big|_0^{\sqrt{3}} \right) =$$

$$= \pi - \left(\sqrt{3} - \underbrace{\arctan \sqrt{3}}_{\frac{\pi}{3}} - 0 \right)$$

$$= \frac{4\pi}{3} - \sqrt{3}$$

4. Funkcijo

$$f(x) = \frac{2}{2+2x-x^2}$$

razvijte v Taylorjevo vrsto okoli točke $a = 1$. Zapišite predpis za koeficiente a_m, ki pripadajo potencam $(x-1)^m$, $m \geq 0$, v Taylorjevem razvoju. Določite vrednosti $f^{(2022)}(1)$ in $f^{(2023)}(1)$.

$$t = x - a = x - 1 \rightarrow x = t + 1$$

$$f(t) = \frac{2}{2+2(t+1)-(t+1)^2} = \frac{2}{2+\cancel{2t}+2-t^2-\cancel{2t}-1} = \frac{2}{3-t^2}$$

GEOM. VRSTA: $\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n$ za $|q|<1$

$$\frac{2}{3-t^2} = \frac{2}{3(1-\frac{t^2}{3})} = \frac{2}{3} \cdot \sum_{n=0}^{\infty} \left(\frac{t^2}{3}\right)^n = \sum_{n=0}^{\infty} \frac{2}{3} \cdot \frac{t^{2n}}{3^n}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{2}{3^{n+1}} (x-1)^{2n}$$

$$m = 2n \rightarrow n = \frac{m}{2}$$

$$a_n = \begin{cases} \dfrac{2}{3^{\frac{n}{2}+1}} & ; n = 2k \\ 0 & ; \text{sicer} \end{cases}$$

$$f^{(2022)}(1) = a_{2022} \cdot 2022! = \frac{2}{3^{1012}} \cdot 2022!$$

$$f^{(2023)}(1) = \underset{\overset{\shortparallel}{0}}{a_{2023}} \cdot 2023! = 0$$