Social Choice and Voting Procedures

CHAPTER 6

> The general will is always right and tends to the public advantage; but it does not follow that the deliberations of the people are always equally correct. Our will is always for our own good, but we do not always see what that is; the people is never corrupted, but it is often deceived.
>
> —Jean-Jacques Rousseau

I. Three Voting Situations

This chapter illustrates the use of axiomatic models by investigating some of the procedures groups of voters use to determine collective judgments from individual preferences. These procedures characteristically have certain injustices associated with them. An axiomatic approach reveals that attempts to redesign the procedures or invent new ones to avoid these inequities are doomed to frustration.

An illustrative real-world example is the U.S. Senate and its attempts to reach agreement on certain types of important issues. The model we develop concerns certain kinds of collective judgments, which are exemplified by the following three illustrations.

Example 1

The president nominates a South Carolina lawyer for a position on the U.S. Supreme Court. The Senate must decide whether to confirm the nomination or not.

Example 2

Three proposals for dealing with the dependents' deduction feature of the federal income tax have been offered. Proposal *A* calls for a substantial increase in the amount of the deduction so that it will more accurately reflect the costs of rearing children in today's economy. Proposal *B* seeks the abolition of the dependents' allowance; its advocates wish to discourage parents from planning large families. Proposal *C* is simply that the present level of the deduction be retained. The Senate must adopt one of these mutually exclusive proposals. This is an example of a *Social Choice Problem*.

Example 3

A commission on national goals asks the Senate for its evaluation of the order of importance of three current problems: the economy, the plight of urban areas, and the protection of the environment. In this situation, the Senate must indicate an *ordering* of three alternatives. This is an example of a *Social Welfare Problem*.

There are 100 senators, two from each state. Assume that there has been sufficient discussion and debate on the matters before the Senate so that each member has already determined his or her own personal preferences among the alternatives open. What procedure should be used in passing from this set of 100 individual preferences to a collective preference?

II. Two Voting Mechanisms

A. Simple Majority Voting

The first situation, that of confirming a nomination to the Supreme Court, poses little difficulty. Each senator announces a vote for or against the nominee. If a simple majority of those voting favors the nominee, the nominee is confirmed. Otherwise, the nominee is rejected. (Should the Senate split evenly, then the ballot of the vice president is counted to Vice President determine the majority position.)

Decision making by simple majority voting is, of course, the most familiar scheme for determining the collective judgment of a group of individual voters. Together with the concept that each individual has but a single vote, it forms the heart of what many would define as *democracy*. "The very essence of democratic government," wrote Alexis De Tocqueville in *Democracy in America* (1835), "consists in the absolute sovereignty of the majority; for there is nothing in democratic states which is capable of resisting it." In his first inaugural address, Abraham Lincoln observed, "Unanimity is impossible; the rule of a minority, as a permanent arrangement, is wholly inadmissible; so that, rejecting the majority principle, anarchy or despotism in some form is all that is left."

As you will see, simple majority voting is a fair and effective procedure to adopt when a group must decide between *two* alternatives or candidates. But there are many situations in which a society needs to make a choice among three or more alternatives. In each of the U.S. presidential elections of 1980, 1992, 1996, 2000, and 2004, there were strong "third-party" candidates who appealed to millions of voters as more attractive than the major party nominees. There are numerous elections for governors, state and county offices, and legislative positions where no single candidate is the first choice of a majority of the voters.

What happens, then, when the group must choose among three or more alternatives? How does the Senate actually reach a decision when faced with a situation like that described in Example 2? It adopts a procedure used by many legislative bodies: change the format of the problem from one involving a choice among three alternatives to a series of choices between two alternatives.

To illustrate this process with the income tax example, the Senate might first decide between proposals A and B on a simple majority vote. The winning proposal would then be pitted against C and the eventual winner then decided by a simple majority vote between these two alternatives.

The idea is to use the simple majority principle—because of a strong belief in its fairness—even when it may not be immediately applicable. Is there anything wrong here?

Re-examine for a moment the individual preferences of the senators. Assume that a certain amount of reasonableness and consistency exists in each senator's personal ordering of the desirability of the three proposals. In particular, assume that each senator's ordering is *transitive*. Transitivity means that if x, y, and z are any three alternatives and a senator prefers x over y and prefers y over z, then the senator must prefer x over z.

If an individual's preferences are transitive, then his preference list can be denoted in a convenient way. Suppose a senator finds proposal C most attractive, proposal B least attractive, and proposal A intermediate to the other two. We may denote the preference list by $(C\,A\,B)$. Then transitivity implies that one proposal is favored over another exactly if it appears to the left of the other in the list.

We now make our first demand on the decision-making process: the collective preference must also be transitive. We want to guarantee that whenever the group prefers x to y and prefers y to z, then it must also prefer x to z. It is on this imminently reasonable and apparently innocent demand that simple majority voting stumbles badly. At least since the time of the Marquis de Condorcet (1743–1794), those concerned about just voting procedures and mechanisms noted the possibility that intransitive social preferences could result when the variation of simple majority voting we described is applied to a list of individual transitive preferences.

To be specific, there are six possible ways an individual can rank-order the three proposals: $(A\,B\,C)$, $(A\,C\,B)$, $(B\,C\,A)$, $(B\,A\,C)$, $(C\,A\,B)$, and $(C\,B\,A)$. Suppose that the preferences of the senators on the dependents' allowance proposals break down as follows:

$$(A\,B\,C)\text{: 31 votes,}\quad (B\,C\,A)\text{: 34 votes,}\quad (C\,A\,B)\text{: 35 votes.}$$

To simplify this example, we assume that none of the other three orderings are represented.

Which proposal will be adopted? If the originally outlined procedure is followed, the Senate will first choose between A and B. Since 66 senators prefer A over B, B will be eliminated from consideration. A second vote will be taken between A and C. Now 69 senators will opt for C, and only 31 for A. Thus, the Senate would adopt Proposal C.

A loud objection can be expected from the advocates of proposal A. It has already been established, they would argue, that the Senate prefers A to B. It is also clear that in a direct vote between B and C, B would receive 65 votes so that the Senate certainly prefers B to C. But if the Senate prefers A to B and B to C, then it must prefer A over C to maintain transitivity.

As you have just seen, the Senate's normal procedures do not necessarily lead to transitive group preferences. But is transitivity always so important? In any legislative situation, it might be argued, the body always has at any moment only the option between two proposals. Only after one of the original two proposals is voted down in favor of the other may a third proposal be introduced.

If group transitivity is not guaranteed, however, more serious problems arise. The result of the legislative deliberation may depend, not on the individual wishes of the members or the inherent worth of the proposals, but on the *order* in which the proposals are offered for consideration. To illustrate with this same example, suppose the agenda is arranged so that A and C are the two original proposals discussed. When a vote is taken, C triumphs over A. When proposal B is finally introduced, it competes against C and wins, 65 to 35.

The author of each of the three proposals A, B, and C then has a legitimate argument that his or her proposal is the one that is "most favored" by the Senate as a whole. Although legislative bodies almost universally employ the modification of simple majority voting we've discussed here, we see that it fails to be a just one. The procedure yields nontransitive group preferences. It does not always produce the same collective preference given the same set of individual preferences. It is subject to manipulations by those who control the ordering of items on the agenda.

A suggested modification of simple majority voting when there are more than two alternatives is to conduct all possible two alternative elections, decide each one by majority vote, and declare as the winner the single alternative that beats all others. Such an alternative, which triumphs over every other choice in head-to-head balloting, is called the *Condorcet Winner*. While this is an appealing decision rule, it doesn't always work, as it is possible that no Condorcet Winner exists. In our Senate example, there is no Condorcet Winner, since C beats A, A beats B, and B beats C.

What procedure should be used if the group wishes to guarantee transitivity and to guarantee that the group decision is purely a function of the individual preferences? How is the "will" of the group to be determined? One possibility often suggested is to adopt a *weighted voting scheme*.

B. Weighted Voting

Weighted voting mechanisms are often used to score athletic, artistic, and beauty contests. The individual ratings of a collection of judges are pooled to determine the final overall rankings of the contestants. Preassigned numerical weights are attached to each first-place rating, each second-place rating, and so on. A contestant receives a score that is the sum of the weights of the opinions of the individual judges. The group ranking of the contestants is then determined by their total scores. The person with the highest number of points is the winner, the individual with the next highest number is the first runner-up, and so on down the list. Notice that this procedure can be employed in a situation either like Example 2 or like Example 3.

Example 4

An instructor offers a class of 25 students the option of a take-home final examination, an in-class final examination, or a major term paper. The class will choose the single option that every student will experience.

Table 6.1 shows how many students rated each option first choice, second choice, and third choice.

Table 6.1

	Take-Home Exam	In-Class Exam	Term Paper
First Choice	7	10	8
Second Choice	12	3	10
Third Choice	6	12	7

Suppose 4 points are given for each first-place vote, 2 for second place, and 1 for third. Then the points for each option and total points are displayed in Table 6.2. The term paper emerges as the winner.

Table 6.2

	Take-Home Exam	In-Class Exam	Term Paper
Points			
4 for First	28	40	32
2 for Second	24	6	20
1 for Third	6	12	7
TOTAL	58	58	59

But suppose a different number of points are designated for each place. If, for example, we give 10 points for each first-place vote, 7 for second, and 3 for third, then a different outcome arises. Table 6.3 shows the results. Here the take-home exam is the overall top choice.

Table 6.3

	Take-Home Exam	In-Class Exam	Term Paper
Points			
10 for First	70	100	80
7 for Second	84	21	70
3 for Third	18	36	21
TOTAL	172	157	171

This example demonstrates the first problem with weighted voting: the outcome may depend on exactly how many points are given for each place.

Is there an allocation of points under which the in-class exam wins? Take 10 points for first, 3 for second, and 1 for third. Table 6.4 shows the results.

Table 6.4

	Take-Home Exam	In-Class Exam	Term Paper
Points			
10 for First	70	100	80
3 for Second	36	9	30
1 for Third	6	12	7
TOTAL	112	**121**	117

There are even more serious weaknesses with a weighted voting scheme: *individual voters may have incentives to falsify their true preferences.*

Consider a beauty contest example in which there are four contestants, labeled w, x, y, and z and three judges. Judges 1 and 2 each rank the contestants in the order $(x\,y\,z\,w)$, while judge 3 ranks them $(z\,w\,x\,y)$. If 5 points are assigned for a first-placed ranking by a judge, 4 points for second, 2 for third, and 1 for fourth, then x earns 12 points, y and z each earn 9, and w earns 6. The winner is contestant x, while y and z tie for second, and w is last.

Suppose that between the time the judges' ratings are submitted and the winner is announced, it is discovered that y has broken the rules of the contest. He is disqualified. The scoring system is now applied to the remaining contestants. It should yield the same results, we believe, especially since y is inferior to x, according to the tastes of each individual judge.

Yet if y is deleted, the rankings become $(x\,z\,w)$ for judges 1 and 2 and $(z\,w\,x)$ for judge 3. Now when the weights are tabulated, x still has 12 points, but z has 13 and w has 8. The master of ceremonies dutifully declares z the winner of the contest. Needless to say, x is furious and his attorney sues the contest committee, claiming her client has been treated unjustly.

This weighted voting scoring mechanism violates an ethical value and poses a practical political problem. Whether a group believes x is better than z or not should be a judgment independent of the group's feelings about a third contestant y. Weighted voting does not preserve this independence.

Here is the practical political problem: In the example, judges 1 and 2 have given their true preferences in their ratings. They think x is best and would like to see x emerge as the eventual winner. Suppose that they have heard rumors that y has not been completely rigorous in following the rules. If judges 1 and 2 were to switch their ratings to $(x\,y\,w\,z)$, they would make it more likely for x to win over z in the event that y is disqualified. These two judges would be falsifying their own preferences.

A fair and equitable voting mechanism should not encourage such falsification. Each voter should feel secure in casting a personal ballot that lists the alternatives exactly in the order in which she would like to see the outcome. Weighted voting schemes remove this security.

The procedure of weighing places in individual preference orderings with numbers and using these numbers to find the societal ordering of proposals or candidates is attributed to Jean-Charles de Borda (1733–1799). His "Mémoire sur les Élections au Scrutin," published in 1781, was the first mathematical theory of elections. When confronted with the possibility that voters might mask their true preferences in order to help their favorite candidate emerge on top, Borda is reported to have replied "My scheme is only intended for honest men."

DEFINITION We call a voting mechanism *manipulable* if there is at least one voter who by disguising his true preferences may ensure a group preference ranking he prefers to the one that would have been obtained had he submitted his true preference ordering.

We have just seen that weighted voting is a manipulable mechanism. The terms *nonmanipulable*, *strategy-proof*, or *sincere* are used to describe voting mechanisms that are not manipulable.

Various other schemes have been proposed for determining a group-preference ranking from lists of individual preferences, and some of them are widely used. These include plurality voting, instant runoff voting, approval voting, range voting, and proportional representation. Each seems to suffer from one or another defect. The injustices of

these voting mechanisms raises the question of whether it is possible to design one that everyone will agree is just and democratic. If it is possible, what would the rules of such a voting procedure look like?

III. An Axiomatic Approach

What is a *just* voting mechanism? To answer this question, we begin by listing some conditions or axioms that a voting system might reasonably be required to satisfy if it is to be labeled a "fair" one. Once the set of axioms is set, we can ask mathematical questions. Is the set of axioms consistent? If so, how many different structures satisfy them? If the axioms are inconsistent, which ones should be eliminated or modified?

In the first place, the mechanism will be translating a list of individual voter preferences into a group-preference list. The voters may differ greatly in their likes and dislikes of candidates or proposals. We do not wish to restrict the freedom of any voter to state her true preferences. Accordingly, the first axiom looks like this:

AXIOM 1 (INDIVIDUAL SOVEREIGNTY) Each voter may order the candidates (or alternative proposals) in any way he or she chooses and may even indicate indifference between pairs of candidates.

The second axiom demands that the system always produce a societal judgment that is transitive and depends only on the individual ballots cast by the voters.

AXIOM 2 (EXISTENCE OF SOCIAL WELFARE FUNCTION) For every collection of lists of individual preferences, the mechanism produces a unique list of society's preferences. The society's preferences are transitive.

Note that Axiom 2 removes the inequities associated with simple majority voting when more than two alternatives are being considered. It also rules out some schemes that do guarantee transitivity. For example, one mechanism might be to put all the individual lists into a hat and draw out at random one of these, which will be designated society's preference list. Since the societal choice corresponds to some particular individual's, it will be transitive. This scheme would not satisfy Axiom 2, because a second implementation of the mechanism (drawing again from the hat) might result in a different outcome. The *uniqueness* feature of the societal list would be violated.

A decision procedure that simply makes the societal outcome the alphabetical listing of the alternatives or one that selects the eldest voter and declares that person's preferences to be the group's preference would be consistent with Axiom 2.

The third axiom is a weak constraint that has generated no controversy among voting-theory experts. It simply asks that in those cases in which everyone prefers x to y, so does the society.

AXIOM 3 (UNANIMITY) If every individual prefers one alternative to another, so does the society.

It would certainly be unreasonable to claim that the society's ranking reflected that of its members if no one agreed with it. In the context of a "Social Choice Function" where all we require is the determination of a winner (society's top choice), the principle of

Unanimity is usually called *Pareto efficiency*. A social choice function is Pareto-efficient if it chooses alternative a whenever a is at the top of every voter's list.

The weighted voting schemes discussed in Section II satisfy Axioms 1, 2, and 3; the proof is left to the reader. The fourth axiom is designed to eliminate the difficulties associated with such systems.

AXIOM 4 (INDEPENDENCE OF IRRELEVANT ALTERNATIVES) The social ordering of any pair of alternatives depends only on the preferences of the individuals between the members of that pair.

This axiom implies that if we want to know whether the society prefers x to y or y to x, we need only examine the relative rankings of x and y on each voter's preference list; we need not look at the rankings of any other candidates.

To understand this axiom better, return for a moment to the beauty contest example. If the rankings as originally turned in by the judges give a group judgment of x higher than z, then any other set of ballots in which judges 1 and 2 rank x higher than z and in which judge 3 ranks z higher than x will result in a group judgment of x higher than z if Axiom 4 holds. In other words, any two elections in which all voters preserve their preference between two particular candidates will yield the same group preference between *those* two candidates.

Let's illustrate this point with an example. The American Film Institute (AFI) selects a panel of three famous critics to choose the best movie ever produced. The critics are Roger, Janet, and Zoey and the three finalist films are *Citizen Kane*, *The Godfather*, and *Casablanca*. Each critic will submit a ranking of the three movies. A social welfare function will then be used to obtain an overall ranking of the three. The exact details of the particular social welfare function are unknown, but the AFI guarantees that it does satisfy Axiom 4.

We'll consider the relative rankings of *Citizen Kan*e and *Casablanca*. Suppose that the individual rankings of the critics are those shown in Ballot 1, and in the overall ranking, *Citizen Kane* winds up above *Casablanca*. If Axiom 4 holds, then *Citizen Kane* would also end up rated higher than *Casablanca* if the individual rankings are those of Ballot 2. This would be true because Roger and Janet rated *Citizen Kane* above *Casablanca* and Zoey listed *Casablanca* higher than *Citizen Kane* in both ballots. No individual voter changed a relative ranking of these two alternatives. Note that we do not know that the social ranking actually put *Citizen Kane* in a higher position than *Casablanca*; that is not material to the question of whether Axiom 4 holds.

Ballot 1

	Roger	Janet	Zoey
(1)	Citizen Kane	The Godfather	Casablanca
(2)	Casablanca	Citizen Kane	The Godfather
(3)	The Godfather	Casablanca	Citizen Kane

Ballot 2

	Roger	Janet	Zoey
(1)	Citizen Kane	Citizen Kane	Casablanca
(2)	The Godfather	Casablanca	The Godfather
(3)	Casablanca	The Godfather	Citizen Kane

A voting mechanism that satisfied Axiom 4 would ensure that voters gain nothing by disguising their true preferences.

It is very easy to design a voting mechanism that satisfies the first four axioms. Simply designate some particular voter as a dictator and decree that society's preference list will just be a copy of that one person's list. The reader should verify that this dictatorial mechanism is consistent with Axioms 1–4. Although having a dictator is certainly an extremely efficient voting mechanism, it is not what most people would call a "democratic" institution. The final axiom rules out such systems.

AXIOM 5 (NONDICTATORSHIP) There is no voter with the power that for all choices x and y, if he ranks x over y, then so does the society regardless of how other voters feel about x **and** y.

A dictator is a voter whose submitted preference list always becomes the society's preference list.

To make the axiomatic model realistic, assume that there are a finite number of individual preference lists. To make it interesting, assume that there are at least three different alternatives to be ranked. (The reader is asked to show that simple majority voting satisfies Axioms 1–5 if there are exactly two candidates or proposals being considered.)

These five axioms describe conditions all of which seem natural and desirable to demand of a voting mechanism. You may, in fact, believe that the axioms demand too little for the mechanism to deserve the adjective "democratic." The axioms do not demand, for example, that each voter's preference list be treated equally; some individuals might be given more "votes" than others. The axioms do not require that society prefer x to y if a simple majority prefers x to y. The axioms also do not insist that the same procedures be used on all pairs of alternatives. Conceivably, a mechanism that used a dictator to decide between Proctor and Swenton while using simple majority voting on Emerson vs. Peterson might be allowed.

The surprising fact is that even this "reasonable" set of axioms is inconsistent. The five demands are incompatible with each other. It is impossible to devise any voting mechanism that will simultaneously satisfy all of them.

This result is known as the General Impossibility Theorem. It was first stated by Kenneth J. Arrow in 1951 in a pioneering essay that sought to place voting theory on an axiomatic basis. Arrow's original proof contained a technical error and a correct proof was first supplied by Julian Blau in 1957. Arrow's Theorem has provoked a considerable amount of discussion by social scientists, philosophers, political theorists, and economists.

IV. Arrow's Impossibility Theorem

We state the theorem in a manner that is both provocative and that indicates the direction of its proof:

THEOREM (ARROW'S GENERAL IMPOSSIBILITY THEOREM) Axioms 1–4 imply the existence of a dictator.

The remainder of this section presents a proof of the theorem. Assume, then, that there is a voting mechanism satisfying Axioms 1–4. We need one additional definition.

DEFINITION A set V of individual voters is *decisive for alternative x against alternative y* if x is socially chosen by the voting mechanism whenever every individual in V prefers x to y and every individual not in V prefers y to x.

This concept is somewhat subtle and requires some explanatory remarks:

a. If the mechanism is a dictatorial one, then the dictator is a one-person set who is decisive for every pair of alternatives.

b. Axiom 3 on Unanimity asserts that the set of all voters is decisive for every pair of alternatives. Should every voter prefer x to y, then so would society. Of course, not every voter might share this preference between x and y.

If some of the voters prefer x to y and others prefer y to x, we need to know more about the details of the voting mechanism to determine the societal ranking.

c. Decisiveness is really a *potential* power. If V is a set that is decisive for x against y, then one of the conditions that must be present in order to predict that society prefers x to y is that everyone in the set V prefers x to y. If a particular individual belongs to V and he prefers y to x, then the fact that V is decisive for x against y does not really give V much influence on the outcome.

d. The other condition that must be met if decisiveness is to be used to predict a societal ranking is that all the individuals not in V must prefer y to x. If V is decisive for x against y, if everyone in V prefers x to y, and if someone not in V also prefers x to y, then we can make no accurate prediction about the societal ranking of x against y unless we have more detailed knowledge about the voting mechanism.

To clarify this point, suppose we have a society with seven members: Mike, Judy, Eli, Sherry, Abby, John, Sasha, and Anne. The voting mechanism is simple, but rather peculiar. The societal ranking is always exactly the opposite of Mike's preference ranking. Let x and y be any two alternatives. Then the set whose members are Eli, Abby, and Sasha is decisive for x against y. If these three prefer x to y and the other four members prefer y to x, then, in particular, Mike prefers y to x. Since Mike prefers y to x, the society prefers x to y. Consider, however, what happens if Eli, Abby, Sasha, and Mike all prefer x to y. Then society will prefer y to x, even though Mike has voted the same way as all the members of a decisive set.

e. A set V may be decisive for x against y but not necessarily decisive for y against x or decisive for any other pair of alternatives. This is due to the fact that Axioms 1–4 do not require the voting mechanism to operate the same way for all pairs of alternatives.

With these warnings about the notion of a decisive set in mind, we proceed to the main part of the proof of the theorem. The proof proceeds by verifying two claims:

Claim I There is some pair of alternatives and some individual who is decisive for that pair.

Claim II If an individual is decisive for some pair of alternatives, then he or she is decisive for every pair of alternatives—that is, the individual is a dictator.

Proof of Claim I For any pair of alternatives x and y, there is at least one nonempty decisive set—namely, the set of all individuals. Among all sets of individuals that are decisive for some pair of alternatives, pick a minimal set. This is a set V of voters and a pair of alternatives x, y so that V is decisive for x against y and no proper subset of V is decisive for *any* pair of alternatives. Axiom III on unanimity means that no empty set can be decisive of any pair of alternatives; thus, V contains at least one voter.

If such a minimal decisive set contains exactly one voter, then we are done with Claim 1. Hence, assume that V contains at least two voters. Let V^* be the set consisting of exactly one voter from V, $V^{\#}$ the subset of V consisting of all voters in V not in V^*, and let V' be the set of all voters in the society not in V. Now V^* is a proper subset of V. We shall show that V^* is decisive for some pair of alternatives, thus contradicting the minimality of V.

Suppose that V is decisive for x against y, and let z be any other alternative. Suppose that the voter in V^* ranks the alternatives $(x\,y\,z)$, all the voters in $V^{\#}$ rank them $(z\,x\,y)$, and all the voters in V' rank them $(y\,z\,x)$.

Note first that all voters in $V = V^* \cup V^{\#}$ prefer x to y and that all voters not in V prefer y to x. Since V is decisive for x against y, society prefers x to y.

Next note that $V^{\#}$ is smaller in size than V, so it is not decisive for any pair. In particular, $V^{\#}$ is not decisive for z against y. This implies that society prefers y to z, for otherwise we would have society preferring z to y when everyone in $V^{\#}$ does and no one outside $V^{\#}$ does.

Finally, use the transitivity of the societal preference. Society prefers x to y and y to z. Thus, society prefers x to z.

We then have one election in which V^* prefers x to z, everyone outside V^* prefers z to x, and the society prefers x to z. By Axiom 4 on Independence of Irrelevant Alternatives, society will prefer x to z *whenever* all individuals maintain these preferences between x and z. Hence, V^* is decisive for x against z. This contradicts the assumption that V is a minimal decisive set. The conclusion, then, is that minimal decisive sets contain precisely one voter. Claim I is verified.

Proof of Claim II Let J be some individual member of the society and write:

1. "$a\overline{D}\,b$" to mean that a is socially preferred to b whenever J prefers a to b regardless of the orderings of other individuals
2. "$aD\,b$" to mean that a is socially preferred to b if J prefers a to b and all other voters prefer b to a

These notations are useful since the condition of dictatorship is that $a\overline{D}\,b$ *for all pairs of alternatives* a and b, while aDb is true if and only if J is a decisive set for a against b.

To complete the proof of Claim II, the following lemma is useful.

LEMMA Suppose there are three alternatives a, b, c. Then

1. aDb implies $a\overline{D}c$, and

2. aDb implies $c\overline{D}b$.

Proof of Lemma Let J rank the alternatives $(a\,b\,c)$ and suppose everyone else ranks b higher than a and c. Since aDb, we conclude that society prefers a to b. Since all individuals prefer b to c, so does society. By transitivity, society prefers a to c. The axiom on Independence of Irrelevant Alternatives asserts that whenever J prefers a to c, so does society, regardless of how the other voters rank c and a. In other terms, $a\overline{D}c$.

To prove that aDb implies $c\overline{D}b$, suppose first that J ranks the alternatives in the order $(c\,a\,b)$ and all other voters rank them $(c\,b\,a)$ or $(b\,c\,a)$. Since aDb, society prefers a to b. By unanimity, society prefers c to a. Transitivity then gives a society preference of c over b. Applying Axiom 4 again, we have $c\overline{D}b$.

This completes the proof of the lemma.

We can now finish the proof of Claim II. Suppose xDy for some pair of alternatives x and y.

Case 1 There are exactly three alternatives: x, y, z.

We must show that $a\overline{D}b$ for all pairs of alternatives—that is,

(1) $x\overline{D}z$ (2) $z\overline{D}y$
(3) $x\overline{D}y$ (4) $y\overline{D}z$
(5) $z\overline{D}x$ (6) $y\overline{D}x$

The proof of (1) follows directly from the lemma with $a = x$, $b = y$, and $c = z$. Similarly, (2) follows from a direct application of the lemma. Now that we know that $x\overline{D}z$, we also have xDz. Now apply the lemma with $a = x$, $b = z$, and $c = y$. The conclusions are that $x\overline{D}y$ and $y\overline{D}z$, giving (3) and (4). The proofs of (5) and (6) are left to the reader.

Case 2 There are more than three alternatives.

Suppose xDy holds and let a and b be any alternatives.

(i) If x and y are the same as a and b, add a third alternative z to x and y and apply the result of Case 1 to show that xDy implies $x\overline{D}y$ and $y\overline{D}x$. Hence, both $a\overline{D}b$ and $b\overline{D}a$ hold.

(ii) If exactly one of a and b is distinct from x and y, add it to x and y to form a triple and apply Case 1.

(iii) If both a and b are distinct from x and y, two steps are needed: First, add a to x and y; obtain $x\overline{D}a$ so that xDa. Second, consider the triple x, a, b; obtain $a\overline{D}b$.

Thus, xDy for *some* x and y implies $a\overline{D}b$ for *all* alternatives a and b. This completes the proof of Claim II and hence the proof of the theorem. ◇

Since Axioms 1–5 are inconsistent as they stand, any attempt to strengthen them—such as demanding that all voters be treated equally—will not remove inconsistency. A voting system that satisfies some of the axioms must violate some of the others. We will not enter here the heated argument as to which is the "best" axiom to modify or discard. The interested reader may follow the debate by consulting the References.

V. The Liberal Paradox and the Theorem of the Gloomy Alternatives

In the years since Kenneth Arrow formulated his Impossibility Theorem, social scientists, political theorists, economists, philosophers, and mathematicians have examined many aspects of the approach he pioneered. They have looked, for example, at how the axioms might be weakened or how they are interrelated to each other or how other fairness criteria might be formulated. In this section, we state and prove several theorems they have discovered that indicate that even more minimal requirements of fairness on a voting mechanism ensure that it is a dictatorial one.

A. The Liberal Paradox

One of the tenets of classical liberalism, extolled for example in John Stuart Mill's famous 1869 essay *On Liberty* is that each individual should be "locally decisive" with respect to a narrowly defined sphere that is that person's private concern. Society should not be able to decide, for example, which religion you must practice or which books you cannot read.

In the context of the problems Kenneth Arrow addressed, Amartya Sen formulated an *Axiom of Minimal Liberalism*: there are at least two individuals each of whom is strongly decisive for some pair of alternatives. A stronger *Axiom of Liberalism* would require that at least one such pair of alternatives exist for every individual, but Sen does not need such an assumption.

Sen discovered and proved the *Liberal Paradox Theorem*: There is no social decision function that satisfies Citizen Sovereignty, Unanimity, and Minimal Liberalism.

We outline a proof. Suppose we have a social decision rule that satisfies Minimal Liberalism. Then there are individuals, call them John and Amy, and alternatives x, y, z, and w such that John is strongly decisive for x and y while Amy is strongly decisive for z and w. We assume, for simplicity, that all four alternatives are distinct.

If our mechanism also satisfies Citizen Sovereignty, then it must deal successfully with a profile that contains these rankings by John and Amy:

John	Amy
...	...
w	y
...	...
x	z
...	...
y	w
...	...
z	x
...	...

Suppose, in addition, that every other individual also ranked w above x and y above z.

Since John is strongly decisive for the pair x and y and John ranked x above y, the mechanism must rank x above y.

If Unanimity is also assumed, then the mechanism ranks w over x and y over z. Hence, our result is that society ranks w above x, x above y, and y above z. Transitivity would require that society rank w above z. But Amy is strongly decisive for the pair w and z, and she ranked z above w; hence, the mechanism must also rank z above w. Thus, Citizen Sovereignty, Unanimity, and Minimal Liberalism force the mechanism to fail the requirement of transitivity that a social choice function must have.

B. Theorem of the Gloomy Alternatives

Recall first Arrow's fourth axiom on Independence of Irrelevant Alternatives (IIA). Suppose we have two different sets L and L^* of rankings by the individual voters where every voter ranks a above b in both L and L^*. The IIA axiom requires that whenever a finishes higher than b under L, then a must finish higher than b under L^*.

We are going to consider a weaker version of IIA that only deals with the case in which alternative a winds up at the top of the social ranking:

DEFINITION Suppose a and b are two alternatives and we have two different sets L and L^* of rankings by the individual voters where every voter ranks a above b in both L and L^*. A social choice function is *monotonic* if whenever a finishes first under L it also finishes first under L^*.

We also want to consider another desirable outcome for a voting mechanism: it should never produce a social preference ranking P such that there is another possible ranking P' that every voter prefers to P. This criterion is a variation of the Unanimity Axiom. A voting mechanism that satisfies this condition is called *Pareto-efficient* or *Pareto-optimal*. Pareto optimality is an important concept with many applications in game theory, engineering, and the social sciences. The term is named after Vilfredo Pareto, an Italian economist (1848–1923) who used the concept in his studies of economic efficiency and income distribution.

In 1977, Eitan Muller and Mark Satterthwaite demonstrated that in situations with more than two alternatives or candidates, insisting on both monotonicity and Pareto efficiency requires accepting a dictatorial mechanism. Let's state their result more explicitly and examine the proof:

The Muller-Satterthwaite Theorem If there are at least three alternatives and a social choice function is both Pareto-efficient and monotonic, then the social choice function is dictatorial.

Proof The following proof is an adaptation of Philip Reny's [2000] argument. Suppose there are N voters numbered $1, 2, \ldots, i, \ldots, N$. Let the preference ranking for voter i be labeled as P_i. Then the collection $P = \{P_1, P_2, \ldots, P_N\}$ is called a *profile*; we'll also use the term *set of ballots*.

There are five steps in the proof of the Muller-Satterthwaite Theorem.

Step 1. Let a and b represent any pair of distinct alternatives (candidates), and let f be a social choice function—that is, given any profile P of rankings, $f(P)$ is a single element in the set of alternatives. You may think of $f(P)$ as the winner of the election as determined by a social choice function that is monotonic and Pareto-efficient.

Consider a profile P where every single voter placed a at the top of the list and b at the bottom. Since f is Pareto-efficient, we would have $f(P) = a$.

Start with voter 1, and think about what would happen if we begin to change that voter's ranking $P1$ by raising b one position at a time. Since f is monotonic, the social choice remains a as long as b is below a in Voter 1's ranking.

Eventually, we would move b to the top of Voter 1's list, where a would now be in second place. Who could be declared the winner with this new profile? If c is any third alternative, then a was above c for every individual's ranking in both the original and the new profile. By the monotonicity property, f can't choose c as the winner. Thus, either b is declared the winner or a remains the chosen alternative.

If a is still the winner, we repeat the same process with voters 2, 3, 4, and so on. We must eventually reach some voter k so that when b rises above a in Voter k's ranking, the social choice function names b the winner. If, to the contrary, a remains the winner after we have gone through every single voter, then we would have a profile in which b is ranked in top position by every voter and a sits in the second position. By Pareto efficiency, a could not be the winner.

Tables 6.5 and 6.6 show the situations immediately before immediately after we raise b above a in Voter k's ranking.

Table 6.5 Before switching a and b for Voter k.

P_1	. . .	P_{k-1}	P_k	P_{k+1}	. . .	P_N	Social Choice
b	. . .	b	a	a	. . .	a	
a		a	b				a
				b	. . .	b	

Table 6.6 After switching a and b for Voter k.

P_1	. . .	P_{k-1}	P_k	P_{k+1}	. . .	P_N	Social Choice
b	. . .	b	b	a	. . .	a	
a		a	a				b
				b	. . .	b	

Step 2. Now let's move alternative a to the bottom of the lists of voters $1, 2, \ldots, k-1$ and to the second from the bottom position for voters $k+1, \ldots, N$. Examine the parallel pictures in Tables 6.7 and 6.8.

Table 6.7 After switching a and b for Voter k.

P_1	. . .	P_{k-1}	P_k	P_{k+1}	. . .	P_N	Social Choice
b	. . .	b	a	.	. . .	.	
		.	b				a
				a		a	
a		a		b	. . .	b	

Table 6.8 After switching a and b for Voter k.

P_1	. . .	P_{k-1}	P_k	P_{k+1}	. . .	P_N	Social Choice
b	. . .	b	b	.	. . .	.	
.		.	a				b
				a		a	
a		a		b	. . .	b	

Let's determine the social choice under the profiles represented by Tables 6.7 and 6.8. Start with Table 6.8 and compare it with Table 6.6. No individual's ranking of b versus any other alternative has changed: b is first for voters $1, 2, \ldots, k$ and last for voters $k+1, \ldots, N$. Since f is monotonic and b was the winner for the profile of Table 6.6, b must also be the winner for the profile of Table 6.8.

Now compare Table 6.8 with Table 6.7. The social choice for Table 6.8 is b and f is monotonic, so the social choice in Table 6.7 could only be b or a. If the social choice for Table 6.7 were b, then monotonicity would also imply that the social choice for Table 6.5 would also have to be b. Since we've already shown that the social choice for Table 6.5 is a, we would have a contradiction. Thus, the social choice for Table 6.7 must be a.

Step 3. Consider now the profile of rankings shown in Table 6.9 where c is an alternative distinct from a and b.

We can obtain the profie of Table 6.9 from the Table 6.7 profile without altering the ranking of a versus any other alternative in any individual's ranking. By monotonicity, the social choice in Table 6.9 must also be a.

Step 4. Now examine the profile of rankings in Table 6.10 that we derived from Table 6.9 by interchanging the ranking of alternatives a and b for voters $k+1, \ldots, N$. This is the only difference between the two profiles.

The social choice for Table 6.9 was a and f is monotonic. Therefore, the social choice for Table 6.10 can only be a or b. We claim that the social choice

Table 6.9

P_1	. . .	P_{k-1}	P_k	P_{k+1}	. . .	P_N	Social Choice
.	. . .	.	*a*	.	. . .	.	
.		.	*c*	.		.	
.	.		*b*	.		.	
.	.						*a*
c	. . .	*c*	.	*c*	. . .	*c*	
b	. . .	*b*	.	*a*	. . .	*a*	
a	. . .	*a*	.	*b*	. . .	*b*	

Table 6.10

P_1	. . .	P_{k-1}	P_k	P_{k+1}	. . .	P_N	Social Choice
.	. . .	.	*a*	.	. . .	.	
.		.	*c*	.		.	
.	.		*b*	.		.	
.	.						*a*
c	. . .	*c*	.	*c*	. . .	*c*	
b	. . .	*b*	.	*b*	. . .	*b*	
a	. . .	*a*	.	*a*	. . .	*a*	

for Table 6.10 must also be a. Suppose, to the contrary, that it was b. Since alternative c is ranked above b in every individual's Table 6.10 ranking, monotonicity would imply that the social choice would remain b even if c were raised to the top of every voter's list, contradicting Pareto efficiency.

Step 5. Observe that alternative a is at the very top of Voter k's list and the very bottom of every other voter's ranking. That means we can built an *arbitrary* profile of rankings with a again at the top of Voter k's preferences without lowering the ranking of a versus any other alternative in any individual's ranking. Monotonicity would now imply that the social choice must be a whenever Voter k puts a at the top of his or her list. Thus, Voter k must be a dictator for alternative a.

Alternative a, however, was chosen arbitrarily, so we can conclude that for every alternative a^* there is some Voter k^* who is dictatorial for a^*. Clearly there cannot be distinct dictators for distinct alternatives, for if $k \neq k^*$, what is the social choice if Voter k lists alternative a first and Voter k^* lists alternative a^* first? Since each is a dictator, the social choice must be a and a^*. But there is only one social choice, so $a = a^*$ and thus $k = k^*$. There is a single dictator for all alternatives.

This completes the proof of the Muller-Satterthwaite Theorem. ◇

We now turn to the question of whether there exist perhaps some more basic principles of fairness in a voting mechanism that might themselves imply both monotonicity and Pareto efficiency.

In our examination of Weighted Voting, we saw this method was subject to *manipulation*: an individual voter might, by falsifying his true preferences, help obtain a social choice he preferred rather than the one that would have resulted had he submitted a ranking that truly reflected his preferences.

Social choice theorists have developed the idea of *strategy-proof* choice mechanisms to characterize systems not subject to such manipulation. We need one bit of notation to state the definition of strategy-proof clearly and concisely.

Suppose that $P = \{P_1, \ldots, P_i, \ldots, P_N\}$ is a profile of rankings and that P_i' is a ranking of Voter i different from P_i. Then (P_i', P_{-i}) denotes the profile obtained by replacing P_i in P with P_i', and $f(P_i', P_{-i})$ is the social choice under this new profile.

The idea of *strategy-proof* is that if a voter submits a ranking P_i' different from the one P_i he prefers and that false submission changes the social choice, then he likes the new choice less than the original choice. Here is the formal definition:

DEFINITION A social choice function f is *strategy-proof* if for every individual voter i and every possible profile P and every possible ranking P_i', $f(P_i', P_{-i}) \neq f(P)$ implies that individual i ranks $f(P)$ above $f(P_i', P_{-i})$ under P_i.

Recall also that the Unanimity axiom implies that each possible candidate could win the election (that is, finish at the top of the social ranking) if all voters ranked that candidate at the top of their individual preference lists. A condition apparently weaker than Unanimity is that for each candidate, there must be some set of individual preferences under which that candidate wins. This condition is called the *onto* property.

DEFINITION A social choice function is *onto* if for each alternative a there is at least one profile under which the social choice is a.

Our next theorem demonstrates that any onto, strategy-proof voting mechanism will automatically be monotonic and Pareto-optimal.

Reny's Theorem If a social choice function is strategy-proof and onto, then it is Pareto-efficient and monotonic.

Proof We follow closely here the proof by Philip Reny [2001]. We break the argument down to three steps.

Step 1. Suppose that the social choice for some profile P is alternative a, and that for every alternative b, the ordering P_i' ranks a above b whenever P_i does for some fixed voter i.

We want to show that $f(P_i', P_{-i})$ is also a. We will proceed using proof by contradiction.

Assume that $f(P_i', P_{-i}) = b$ for alternative $b \neq a$. Since f is strategy-proof, it must be true that P_i ranks a above b. On the other hand, alternative a's ranking does not fall in replacing P_i with P_i'. Hence, a, which was the choice under P,

must also be ranked above $b = f(P_i', P_{-i})$ in P_i'. But this result contradicts strategy-proofness. Thus, $f(P_i', P_{-i}) = f(P) = a$.

Step 2. In this step, we'll demonstrate that the social choice procedure is monotonic. Suppose now that the social choice for some profile P is alternative a, and that for every alternative b, the ordering P_j' ranks a above b whenever P_i does for *every* voter i. Now we can move from $P = (P_1, P_2, \ldots, P_N)$ to $P' = (P_1', P_2', \ldots, P_N')$ by changing the ranking of each voter i from P_i to P_i' one at a time, and because we have shown that the social choice must remain unchanged for every such change, we must have $f(P') = f(P)$. Thus, f is monotonic.

Step 3. In the final step, we'll show that the social choice procedure is Pareto-efficient. Let a be any alternative. Since f is onto, there is some profile P such that $f(P) = a$. Since f is monotonic, the social choice remains alternative a when a is raised to the top of every individual's ranking. But f being monotonic also implies that the social choice remains a regardless of how the alternatives below a are ranked by each individual. Consequently, the social choice is a whenever every individual ranks a at the top—that is, f is Pareto-efficient. Hence, Reny's Theorem is true.

We complete this section with what I call the *Theorem of the Gloomy Alternatives*: if we demand even the simplest conditions on a voting mechanism, then we must settle for a dictator or a manipulable system. Allen Gibbard and Mark Satterthwaite independently discovered and proved this theorem in the mid-1970s.

The Gibbard-Satterthwaite Theorem If there are at least three alternatives and the social choice function is strategy-proof and onto, then the social choice function is dictatorial.

Proof With the given hypotheses, the Reny Theorem tells us that the social choice function is Pareto-efficient and monotonic. The Muller-Satterthwaite Theorem then implies that it is dictatorial. ◇

VI. Instant Runoff Voting

In the light of Arrow's Impossibility Theorem and the results of Gibbard and Satterthwaite, how should we proceed to derive a group decision out of individual preference rankings when we're faced with more than two alternatives?

In this section, we will investigate one alternative, *Instant Runoff Voting*, or IRV for short, that is gaining in popularity. Instant Runoff Voting is a variation of what we might call *Classic Runoff Voting*. Classic Runoff Voting deals with a two-stage process wherein all voters at each stage indicate a single candidate as their top choice. If none of the candidates achieves a majority of the votes, there is a runoff election between the two top candidates.

The 2008 U.S. Senate race in Georgia provides a recent example. In the general November election, the incumbent Republican Senator Saxby Chambliss was the leading candidate with 49.8% of the votes. His Democratic opponent Jim Martin garnered 46.8%.

Allen Buckley of the Libertarian party won 3.4%; two other write-in candidates received a handful votes. The total vote was 3,752,577. A month later, a runoff election between Chambliss and Martin took place. Chambliss won with 57.4% of the 2,137,956 votes cast. Note that voter turnout for the runoff contest in December was substantially smaller than the original numbers for the November election.

In Section I, we saw that runoffs between the top two alternatives may result in situations in which a majority of the voters actually prefer the eliminated third candidate to the winner of the runoff. This sort of result will occur because modified simple majority voting does not always guarantee transitive results. In addition to the theoretical shortcomings, classical runoff voting also poses a number of practical difficulties. Runoff elections are expensive for a city or state to conduct. At the local level a municipal election may cost several hundred thousand dollars; for a state, the outlay may be in the millions of dollars. The candidates must also raise additional money to campaign for an additional month or 6 weeks after the first balloting. Voters must wait that additional period before knowing who will represent them as the winner. Finally, runoff elections typically attract far fewer voters than the first run. In the Georgia example, almost half of those who voted in the initial contest failed to cast a ballot in the runoff.

Instant runoff voting solves many of these problems. Under IRV, each voter submits an individual preference list ranking all the candidates rather than simply indicating a first choice. Then first-place choices are tabulated. If a candidate receives a majority of first choices, that candidate is elected. If no candidate receives a majority of first choices, the candidate receiving the fewest first choices is eliminated. Ballots cast for the eliminated candidate are now counted toward those voters' second choices.

This process continues until one candidate receives a majority and is elected. Once all the preference lists have been submitted, a computer can easily carry out the successive rounds of transferring of votes until a majority winner is found. The election result can be determined shortly after the polls close, with no need of additional campaigning or for additional trips by voters to the ballot box. Thus, Instant Runoff Voting does solve the practical problems that plague classic runoff election processes.

To see in more detail how Instant Runoff Voting works, consider the following example. Suppose we have six different groups of voters of different sizes and four candidates Marc, Rhonda, Julie, and Brian. There are a total of 1,150 voters; a candidate must tally more than 575 votes to win. Table 6.11 displays the voter groups, their sizes, and their shared preferences.

Thus, Marc has 400 first-place votes, Rhonda has 300, Julie has 250, and Brian has 200. Under plurality voting, Marc would win the election. Under classic runoff voting, there would be a second-round election between Marc and Rhonda. In that contest, the 450

Table 6.11 Initial Preference Rankings for IRV Example

Group	I	II	III	IV	V	VI
Size	400	150	150	250	150	50
1st	Marc	Rhonda	Rhonda	Julie	Brian	Brian
2nd	Rhonda	Julie	Marc	Brian	Julie	Julie
3rd	Julie	Brian	Julie	Rhonda	Rhonda	Marc
4th	Brian	Marc	Brian	Marc	Marc	Rhonda

voters in Groups I and VI would vote for Marc while the 700 voters in Groups II, III, IV, and V would opt for Rhonda, making Rhonda the winner.

Instant runoff voting produces a different result. There is no majority winner in Round One, so candidate Brian is eliminated, as he got the lowest number of first-place voters. The 200 people who listed Brian at the top of their preference lists now have their votes transferred to their second-place candidate. Table 6.12 displays the redistributed ballots at the start of Round Two. In this case, Julie gets all 200 votes.

Now Marc still has 400 votes and Rhonda retains her 300, but Julie now has 450. Julie has the lead, but still lacks of majority. We thus end Round Two with no winner. Rhonda has the fewest votes and we eliminate her, moving her votes with the next highest candidate as we enter Round Three. Julie will get 150 votes from Group II, and Marc gets an equal number of votes from Group III. Table 6.13 shows the redistributed ballots for Round Three.

At this point, candidate Julie has 600 votes and candidate Marc has 550. Julie now has a majority and is declared the winner.

In addition to solving the practical difficulties classic runoff elections pose, Instant Runoff Voting has other appealing features. Some voting theorists argue that IRV satisfactorily address the issue of the "third-party spoiler" or "wasted vote" scenario. This is the problem that frequently occurs in many states that don't have runoffs, but that declare as victor the plurality winner. Here is a typical situation: suppose there are three candidates for governor: a Republican, a Democrat, and a Progressive who is to the left of the Democrat on the political spectrum. None of these three is the favorite of a majority of the voters. Each voter can cast only one ballot for a single candidate.

To be specific, suppose the electorate falls into three categories as shown in Table 6.14:

Table 6.12 Redistributed Ballots for Round Two

Group	I	II	III	IV	V	VI
Size	400	150	150	250	150	50
	Marc	Rhonda	Rhonda	Julie	Julie	Julie
	Rhonda	Julie	Marc	Rhonda	Rhonda	Marc
	Julie	Marc	Julie	Marc	Marc	Rhonda

Table 6.13 Redistributed Ballots for Round Three

Group	I	II	III	IV	V	VI
Size	400	150	150	250	150	50
	Marc	Julie	Marc	Julie	Julie	Julie
	Julie	Marc	Julie	Marc	Marc	Marc

Table 6.14

Group	I	II	III
Size	44%	36%	20%
	Republican	Democrat	Progressive
	Democrat	Progressive	Democrat
	Progressive	Republican	Republican

Some of the Group III voters simply prefer the Progressive candidate to the Democrat, while others are also seeking to build the Progressive Party by demonstrating it represents a significant part of the electorate. Early public opinion polls show 44% support for the Republican, 36% for the Democrat, and 20% for the Progressive. In response to the polls, many Group III members fear they would be "wasting" their votes by putting the Progressive at the top of their lists. Their anxiety is that doing so would give the victory to the Republican, their last choice. Similarly, pressure begins to build on the Progressive candidate to drop out of the race. The Democrats say the Progressive is a "spoiler": he can't win and by staying in the race prevents the Democrat, who is the first or second choice of every voter, from winning. History shows that in such situations, most of the Group III voters will cast their ballots for the Democrat in the hope perhaps of electing "the lesser of two evils." The Progressive winds up with only one or two percent of the vote. Political pundits then conclude that there is only very minimal support for a Progressive agenda when, in fact, about one in five people favor it.

Adherents of the Instant Runoff Voting system contend that under their system, the Group III voters can submit their true preferences without fear of hurting their second choice's chances of ultimately winning if their first choice is eliminated. If their third party favorite, the Progressive, does tally 20% of the first-place votes in the initial round, then the party will be taken more seriously by the public, Progressives will be included in future candidate debates, the media will pay more attention to them, and the party might attract more supporters. In a similar fashion, any small third party might benefit from IRV.

In a short 1871 paper "Application of Mr. Hare's System of Voting to the Nomination of Overseers of Harvard College" in the *Journal of Social Science*, the American architect William Robert Ware first introduced Instant Runoff Voting in the form we have described. He was building on an idea of the Australian Thomas Hare, who proposed a related method, called the *single transferable vote*, for electing multiple members to a governing board in a manner that reflected proportional representation among many different constituencies.

Since its first use in Australia at the turn of the 20th century, Instant Runoff Voting has spread to a number of other countries. IRV is used to elect members of the Australian House of Representatives, the President of Ireland, the national parliament of Papua New Guinea, and the Fijian House of Representatives. The Labour and Liberal Democrat parties in the United Kingdom use IRV to select their leaders.

In recent years, IRV has gained much attention in the United States and has been adopted for various elections in many states, including Arkansas, California, Colorado, Florida, Illinois, Louisiana, Maryland, Massachusetts, Michigan, Minnesota, New Mexico, North Carolina, South Carolina, Vermont, and Washington. Instant runoff voting is sometimes called *alternative voting* in the United Kingdom, *preferential voting* in Canada and Australia, and *ranked choice voting* in the United States.

Californians for Electoral Reform summarizes the major arguments in favor of IRV, claiming that it

a. Results in majority rule

b. Eliminates the "spoiler" dilemma, wherein voting for a weak favorite candidate causes one's least favorite candidate to win

c. Allows for a diverse candidate field while also ensuring that the winner has the support of a majority coalition

d. Encourages positive campaigns, because candidates depend on the second choices of voters for other candidates

e. Works cheaply and conveniently, because it collects all the information necessary to determine a majority winner on one ballot

The California group also argues that by ranking candidates, "voters are able to express their true preferences without worrying about wasting their votes or spoiling the election and helping elect their least favorite candidate. For this reason alone, IRV often leads to higher turnout and stronger democracy. Candidates need to build a base of first choice support, but also reach out to the broader voting population in order to be acceptable to the majority."

With so many arguments in favor of IRV, are there good arguments against it? Arrow's Impossibility Theorem implies that IRV cannot satisfy Axioms 1–5. As an illustration of what can go wrong with IRV, let's examine an example with 21 votes whose preferences among four candidates split into four groups whose sizes and numbers are shown in Table 6.15.

The winner needs a majority of 11 or more votes to win. At the conclusion of Round 1, we that Karzi has 7, Barak has 6, Chavez has 5, and Patel has 3. Karzi has the plurality, but falls short of a majority, so Patel is eliminated, and the first choice votes of Group IV members are transferred to Chavez to start Round 2. Table 6.16 shows the resulting situation: Karzi has 7 votes, Barak has 6, and Chavez has 8.

There's still no candidate with a majority. The IRV rules require that we eliminate Barak, redistribute the votes of Groups III and IV, to their next choice and go on to Round 3. Table 6.17 displays the result.

At this point, Karzi has 13 and Chavez has 8. Karzi commands the majority and is declared the winner.

Note that Karzi won, despite the fact that all the voters in Group IV originally ranked Karzi at the very bottom of their lists. To see what difficulties are associated with IRV, examine what should happen if these voters had a change of heart just before the election and moved candidate Karzi to the very top of their preference lists. We expect that Karzi

Table 6.15

Group	I	II	III	IV
Size	7	6	5	3
1st	Karzi	Barak	Chavez	Patel
2nd	Barak	Karzi	Barak	Chavez
3rd	Chavez	Chavez	Karzi	Barak
4th	Patel	Patel	Patel	Karzi

Table 6.16

Group	I	II	III	IV
Size	7	6	5	3
	Karzi	Barak	Chavez	Chavez
	Barak	Karzi	Barak	Barak
	Chavez	Chavez	Karzi	Karzi

Table 6.17

Group	I	II	III	IV
Size	7	6	5	3
	Karzi	Karzi	Chavez	Chavez
	Chavez	Chavez	Karzi	Karzi

Table 6.18

Group	I	II	III	IV
Size	7	6	5	3
1st	Karzi	Barak	Chavez	Karzi
2nd	Barak	Karzi	Barak	Patel
3rd	Chavez	Chavez	Karzi	Chavez
4th	Patel	Patel	Patel	Barak

Table 6.19

Group	I	II	III	IV
Size	7	6	5	3
1st	Karzi	Barak	Barak	Karzi
2nd	Barak	Karzi	Karzi	Barak

should still win, since Karzi is at least as highly rated on everyone's ballot as she was originally. Table 6.18 shows the revised preference lists.

In Round 1, Karzi now has 10 votes, which is not quite a majority, so IRV dictates that we must eliminate Patel, who didn't get any first-place votes, and then Chavez.

After transferring the appropriate votes, we see (Table 6.19) that Karzi retains the 10 votes, but candidate Barak now has 11, which is a majority. Barak wins. This outcome seems perverse: how could Karzi lose an election if more people rank her first? This problem with IRV is often described as the More-Is-Less Paradox: If the winner were ranked higher by some voters, all else unchanged, then another candidate might have won.

Note also that with the original preference rankings, a majority of voters prefer Barak to Karzi, a majority prefer Barak to Chavez, and a majority prefer Barak to Patel, yet Karzi won under IRV. Thus, IRV can lead to what sometimes called the Thwarted Majorities Paradox: a candidate who can defeat every other candidate in a direct-comparison majority vote may not win the election!

We'll illustrate one more problem of IRV, illustrated by an example that comes from Peter Fishburn and Steven Brams. Imagine a municipal election with three candidates: Bitt, Huff, and Wogg. Two of the voters, Mr. and Mrs. Smith, are on the way to the polls when their car breaks down; they are thus unable to register their preferences. Both of them favored Bitt to Huff to Wogg and would have turned in that ranking. "Although they liked Mrs. Bitt best," write Fishburn and Brams, "they were almost as fond of Mr. Huff, but disliked and mistrusted Dr. Wogg."

When the votes were tabulated the next day, it was discovered that 1,608 people had turned in preference lists. Table 6.20 shows the number received for each of the six possible orderings.

Table 6.20

Group	I	II	III	IV	V	VI
Size	417	82	143	357	285	324
1st	Bitt	Bitt	Huff	Huff	Wogg	Wogg
2nd	Huff	Wogg	Bitt	Wogg	Bitt	Huff
3rd	Wogg	Huff	Wogg	Bitt	Huff	Bitt

A candidate needs 805 votes to win. The preference lists submitted show 499 votes for Bitt, 500 votes for Huff, and 609 for Wogg. IRV then requires that we eliminate Bitt and transfer her votes to the other candidates. The 417 votes of Group I go to Huff, making his total 917, and Group II's 82 votes go to Wogg, increasing his total to 691. Huff is the winner. When the Smiths read the result in their newspaper, "they were delighted that Dr. Wogg had not won. They did feel a twinge of regret that their friend, Mrs. Bitt, was beaten. Perhaps their votes would have made a difference."

They certainly would have made a difference, but not the way the Smiths hope. Had they made it to the polls in time, Mrs. Bitt would have had 501 votes at the end of the first round. IRV would have eliminated Mr. Huff, who only had 500. When Huff's votes are transferred, Group III's 143 go to Bitt, giving her a new total of 644, but all of Group IV's 357 votes would go to Wogg, raising his total to 966, well above that required for a majority. The Smiths' well-intentioned votes to help Bitt or Huff would have backfired and made Wogg the winner!

Fishburn and Brams dub this particular problem of IRV the No-Show Paradox: The addition of identical ballots with a particular candidate ranked last may change the winner from some other person to that particular candidate.

VII. Approval Voting

While Instant Runoff Voting has its strong adherents and a number of cities and states have adopted this mechanism, it also has strong critics who are eager to point out some of its potential pitfalls. Peter Fishburn and Steven Brams created the examples we have just seen, associated with the fictional town of Bramburn, that demonstrate some of these problems.

What method of social choice do Fishburn and Brams advocate? They propose an entirely different option for voters. In traditional voting, each individual can only indicate her *single top choice*. For the Borda count, Instant Runoff Voting, or the more general schemes envisioned by Arrow, each voter submits *a ranked ordering* of all candidates. Fishburn and Brams suggest giving each voter the option of voting for any number of

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FIGURE 6.1 The logos for two advocacy groups.

Table 6.21

Group	I	II	III	IV	V	VI
Size	17	40	40	20	46	4
1st	Ford	Ford	Olds	Olds	Saab	Saab
2nd	Olds	Saab	Saab	Ford	Ford	Olds
3rd	Saab	Olds	Ford	Saab	Olds	Ford

candidates for a given office. The candidate who collects the most votes wins. This system is called Approval Voting. In plurality voting, the direction "Vote for one candidate" would appear above the list of contenders. In Approval Voting, the direction would be "Vote for as many candidates as you like."

To illustrate how Approval Voting works and how it may result in outcomes different from Instant Runoff Voting or Plurality Voting, consider the rankings displayed in Table 6.21 for an election among three candidates (Ford, Olds, and Saab) where there are 167 votes, falling into six different groups.

Under traditional plurality voting, Olds wins 60 first-place votes, followed by Ford with 57 and Saab with 50. With Instant Runoff Voting, we would eliminate Saab and transfer 46 votes to Ford and 4 votes to Olds. After the transfer, Ford has 103 votes and Olds has 64, making Ford the winner.

To determine the outcome under Approval Voting, we need to know some additional information. Suppose that all voters check off the names of their top two candidates. Then Ford wins 57 votes from Groups I and II, while picking up an additional 66 votes from Groups IV and V; Ford's total is 123. Now Olds has $17 + 40 + 20 + 4 = 81$ votes. In this case, Saab gets 40 votes from Group II, 40 from Group III, 46 from Group V, and 4 from Group 4. Here 130 voters listed Saab, so Saab is the winner.

For a different scenario, suppose 25 members of Group II list both Ford and Saab, while 15 just list Ford, everyone in Group VI only lists Saab, and everyone in Group I lists both Ford and Olds. In addition, suppose that each of the remaining groups splits in two, half listing their top choice only and half listing their top two choices.

Table 6.22 then shows the number of votes each candidate receives from members of each of the groups. Saab remains the winner with 95 votes.

Another way to tabulate votes under approval voting is to list all the possible subsets of candidates and the number of ballots for each of these. For the example we have just been looking at, we have

Ford	15
Olds	$20 + 10 = 30$
Saab	$4 + 23 = 27$
Ford, Olds	$17 + 10 = 27$
Ford, Saab	$25 + 23 = 48$
Olds, Saab	20
Ford, Olds, Saab	0

Table 6.22

Group	I	II	III	IV	V	VI	Total
Ford	17	40	0	10	23	0	90
Olds	17	0	40	20	0	0	77
Saab	0	25	20	0	46	4	95

Table 6.23

Subset	Ballots	Subset	Ballots
None	1, 100	Olds, Saab	1425
Ford	10, 738	Olds, Dodge	1824
Olds	6561	Saab, Dodge	608
Saab	7626	Ford, Olds, Saab	148
Dodge	8521	Ford, Olds, Dodge	5605
Ford, Olds	3578	Ford, Saab, Dodge	143
Ford, Saab	659	Olds, Saab, Dodge	89
Ford, Dodge	6679	All	523

Source: From Steven J. Brams and Jack H. Nagel, "Approval Voting in Practice," *Public Choice* **71** (1991): 1–17.

From this table, we calculate the total for each candidate:

$$\text{Ford} : 15 + 27 + 48 = 90$$
$$\text{Olds} : 30 + 27 + 20 = 77$$
$$\text{Saab} : 27 + 48 + 20 = 95$$

Advocates for Approval Voting claim that it:

- Is simple
- Is easy to understand
- Is practical to implement
- Increases voter turnout
- Helps elect the strongest candidate
- Gives voters more flexibility
- Gives minority candidates their proper due

Table 6.23 shows another example with four candidates. The data represent actual votes cast under Approval Voting in the 1988 president election of the Institute of Electrical and Electronic Engineers (IEEE). The IEEE is an international organization with more than 200,000 members. In the 1988 poll, 55,310 members returned ballots. We've changed the names of the candidates to Ford, Olds, Saab, and Dodge.

Table 6.24

Group	Ford	Olds	Saab
80	X		X
15		X	X
5		X	X
	80	20	100

Table 6.25

Group	I	II	III
Size	80	15	5
1st	Ford	Olds	Saab
2nd	Saab	Saab	Olds
3rd	Olds	Ford	Ford

To determine a candidate's total, we need to add the votes for all subsets to which that candidate belongs. For example, to find Ford's total:

$$10{,}738(\text{Ford}) + 3{,}578(\text{Ford, Olds}) + 659(\text{Ford, Saab}) + 6{,}679(\text{Ford, Dodge}) + 147(\text{Ford, Olds, Saab}) + 5{,}605(\text{Ford, Olds, Dodge}) + 143(\text{Ford, Saab, Dodge}) + 523(\text{All}) = 28{,}073$$

Similar calculations for the remaining three candidates gives these totals:

Olds: 19,753

Saab: 11,221

Dodge: 23,992

One disadvantage of Approval Voting over Instant Runoff Voting is that voters have no way of indicating a strong preference for one candidate and a weaker one for another candidate if they are willing to accept either of them as an eventual winner. Real voters almost always will have different degrees of support for different candidates. Approval Voting forces individuals to cast equally weighted votes for candidates whose names they check off on the ballot.

Consider, for example, a set of 100 ballots submitted in an Approval Voting situation. Table 6.24 displays the results that the election officers would see at the end of the day. Eighty voters approve of Ford and Saab, 15 approve of Olds and Saab, and 5 approve of Olds and Saab. Thus, Ford gets a total of 80 votes, Olds gets 20, and Saab gets 100. Approval Voting makes Saab the winner.

Of the 80 voters who checked off the names Ford and Saab, we have no idea how many preferred Ford to Saab, how many had the opposite ranking, or how many might have been indifferent between the two. Suppose it were possible to see the preference rankings of our voters? Table 6.25 shows a possible set of rankings; each voter opted to check off the names of his top two choices.

We see from Table 6.25 that candidate Ford was the top choice of an overwhelming majority (80%) of the voters. Surely, any reasonable voting process should give the victory to candidate Ford. Note that candidate Ford wins under Plurality Voting, under Instant Runoff Voting, and wins in a sequence of two candidate Simple Majority contests. Approval Voting has a serious deficiency: a candidate who is the first choice of a staggering majority of voters might not win the election!

The Center for Voting and Democracy finds this flaw in Approval Voting to be a serious one. The Center also notes that Approval Voting does not solve the spoiler problem. Voting for your second choice candidate can in some cases lead to the defeat of your favorite candidate. In the example shown in Tables 6.24 and 6.25, if 55 of the 80 voters in Group I had checked only Candidate Ford and not displayed approval for Saab, then Ford would have won.

As a practical consequence, each candidate might benefit by encouraging her supporters to "bullet" vote—that is, only check her name as acceptable. If everyone did this and all voters complied, then Approval Voting reverts back to Plurality Voting.

Despite some of its theoretical shortcomings, Approval Voting is gaining acceptance as a new societal decision making method especially as an alternative to Plurality Voting. Approval Voting was used as early as the 13th century to select the doge (chief magistrate) of the Venetian Republic. It began to be studied seriously by voting theorists, economists, political scientists, and operations researchers in the middle 1970s when five scholars rediscovered it independently. Robert J. Weber of Northwestern University coined the term "Approval Voting." The selection of the Secretary General of the United Nations uses Approval Voting, and variations of it have occurred in elections in the Soviet Union, 19th-century England, and the American colonies in the 17th century. Some presidential straw polls and statewide referenda in the United States have also employed Approval Voting. Several major professional mathematical societies (Mathematical Association of America, American Mathematical Society, Institute for Operations Research and Management Sciences, American Statistical Association) use Approval Voting to choose officers, as do the Public Choice Society, the Society for Judgment and Decision Making, the International Joint Conference on Artificial Intelligence, the Econometric Society, and the National Academy of Sciences.

Instant Runoff Voting and Approval Voting are both seen as superior to Plurality Voting and appear to be the leading candidates to replace it. There is no agreement concerning which is superior. IRV supporters are quick to point out the shortcomings of Approval Voting, whose adherents are equally swift in noting that IRV fails some fairness criteria. You can follow some of the debate along with the successes and failures having one of these methods adopted by governments by checking their respective websites: www.FairVote.org and www.ApprovalVoting.org. The arguments can become quite heated at times: the mathematician Donald Saari, who has studied voting procedures extensively, once described Approval Voting as an "unmitigated disaster."

VIII. Topological Social Choice

A. Topological Social Choice

In previous chapters, we have often presented *discrete* and *continuous* models of the same situation. Our view of the social choice problem so far has been a discrete one: a finite

collection of voters examining a finite set of candidates. In this section, we will briefly discuss continuous analogs. In particular, we'll look at situations with a finite number of voters but a continuum of alternatives. The techniques and results in this field, *topological social choice*, were pioneered by the mathematician and economist Graciela Chilchilinksy. In this section, we will present a brief nontechnical discussion of topological social choice theory and some of its major findings so far. The field is in active development with new results unveiling connections previously hidden. Proofs of the principal theorems generally require tools and techniques of algebraic topology and thus lie beyond the mathematical prerequisites of this volume.

The fundamental problem of social choice theory is how to aggregate a collection of individual preferences among a set of alternatives into a single collective choice by a procedure that satisfies a number of prescribed fairness conditions. In our discussions so far, we have only examined situations involving a finite group of individuals (the *voters*) and a finite field of alternatives (the *candidates*). Our main result, Kenneth Arrow's Impossibility Theorem, deals with this discrete framework.

As an example, consider Dave and Judy's selection for a site to build their dream home. They have just purchased a large piece of property that contains a circular lake of radius 1 mile. The couple agree that they want their new house to be located on the lakeshore but disagree as to where the location should be. Four possible sites A, B, C, and D have been identified; they are shown in Fig. 6.2.

We want to identify some choice rule that will take Dave and Judy's preferences and select one of the four sites. For simplicity, let's assume that Dave and Judy are only asked to submit their top choice, the site each likes the best. What are some conditions we might want to impose on choice rules? We certainly want the rule to handle any combination of top choices submitted by our two voters and to output one of the four possible sites. We might also want to restrict our choice rule by insisting that it satisfy three additional constraints:

Unanimity: If all the voters choose the same site, then the choice rule also picks this site.

Anonymity: The choice rule treats all voters impartially. The same collection of profiles—whether they have the voters' names on them or not, whether we change the names of the voters on the ballots or not—always produces the same outcome. The choice rule pays attention only to the profiles submitted, not to which voter turned in which profile.

Stability: If one voter changes her opinion and now claims that her favorite site is next to her previous choice, then the output of the choice rule changes at most to a site next to the previous output—that is, choice rule selects the same site it did before or to one adjacent to it.

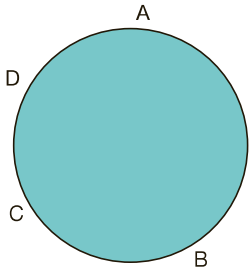

FIGURE 6.2

Table 6.26

		DAVE'S	TOP	CHOICE	
		A	**B**	**C**	**D**
JUDY'S	A	*A*	*A*	*B*	*A*
TOP	B	*A*	*B*	*C*	*D*
CHOICE	C	*B*	*C*	*C*	*C*
	D	*A*	*D*	*C*	*D*

Thus, stability implies that if the choice rule originally chose B when Judy submitted D and David submitted A, then the choice rule will choose A, B, or C if Judy submits A or C and David still submits A. The stability condition is meant to ensure that the social choice doesn't move much under small "errors" of the voters. Table 6.26 shows the assignments of choice rule that has stability.

Note that the ways we have formulated these conditions of unanimity, anonymity, and stability do not restrict the number of voters or the number of alternatives. They could equally well apply to a social rule that picks the best of n possible home sites taking into account the preferences of k family members.

It is an interesting exercise to show that in the case of Dave and Judy, it is not possible to construct a social choice rule that satisfies stability, anonymity, and unanimity if there are five sites available. In fact, Yuliy Baryshnikov (1993) proved a more general theorem that if $n > 2k$, there is no stable, anonymous, unanimous social choice rule.

To move to a *continuous* model of social choice, suppose Dave and Judy are free to choose any spot along the shore to locate their dream home. Each of their first-place choices is then a point on a unit circle; the social choice rule must look at their choices and assign some point on that circle. Set up a standard coordinate system with the origin at the center of the lake. Note that we can identify each point P on the circle with the directed line segment from the origin to P. This vector has length 1 and defines an angle θ between the positive-horizontal axis and the vector, measured in a counterclockwise fashion. We can thus describe the location of a point on the unit circle by giving its pair of Cartesian coordinates or simply by stating the angle θ. Dave and Judy each have infinitely many choices for θ. See Fig. 6.3.

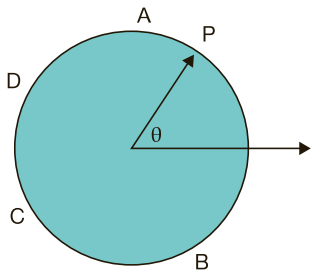

FIGURE 6.3

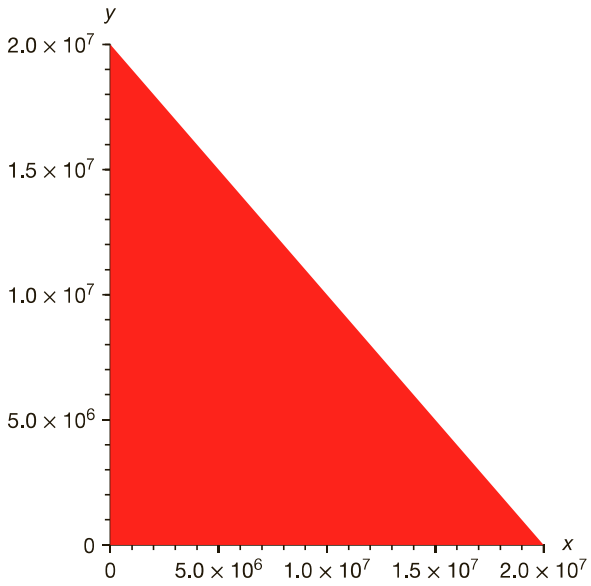

FIGURE 6.4

There are many situations in which the society needs to make choices among infinitely many possibilities. Consider for example a city council allocating a budget among different departments. For a very simple case, suppose the council has decided it can spend up to \$20 million and wishes to split the budget between education (E) and municipal services (M). The council may then choose any pair of nonnegative numbers E and M whose sum does not exceed \$20 million. The set of possible choices is equivalent to the set of points (x, y) in the first quadrant whose sum is less than or equal to 20,000,000—that is, $P = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 20,000,000\}$. Fig. 6.4 illustrates this set.

In reality, the city council must distribute its budget over many departments. If there are n departments, then the set of all possible choices is equivalent to the set of points in Euclidean n-dimensional space all of whose coordinates are nonnegative and that sum to no more than 20,000,000—that is, $P = \{(x_1, x_2, x_3, \ldots, x_n)$: each $x_i \geq 0, x_1 + x_2 + \cdots + x_n \leq 20,000,000\}$.

In a series of papers beginning in the 1980s, Graciela Chichilnisky introduced a *topological* approach to social choice theory—that is, an approach anchored in the concept of *continuity*. Informally speaking, we assume first that if an individual prefers alternative x to alternative y, then that person prefers alternatives that are sufficiently close to x to alternatives close to y. We also want to consider social choice rules that follow a similar rule: if two profiles of preferences are sufficiently close together, then the social choices from the two profiles should also be close. Note that we will need some precise way to talk about "close" and "sufficiently close."

In beginning calculus, we study continuous functions between sets of real numbers. In more advanced classes, we investigate continuous functions between subsets of Euclidean spaces; we may, for example, examine a function that assigns a point in

Euclidean three-dimensional space E^3 to each point of the plane or perhaps a function that assigns an n-dimensional vector to each m-dimensional vector. A function f that assigns an element of a set B to every element of a set A is called a *continuous function from A to B* if, roughly speaking, every pair of sufficiently close points in A are sent to a pair of very close points in B. More exactly, suppose we have distance measures d_A and d_B for the sets A and B, respectively. The function f is *continuous at a point* x in A if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $d_B(f(x), f(y)) < \varepsilon$ whenever $d_A(x, y) < \delta$. We say that f *is continuous on A* if it is continuous at each point of A.

A *neighborhood* of an element p is the set of elements whose distance from p is less than r for some positive number r, called the *radius* of the neighborhood. An equivalent definition of continuity is that for every neighborhood V of $f(p)$, there is a neighborhood U of p such that every element of U is sent by f to some element of V. The mathematical discipline *topology* studies continuity in Euclidean spaces of all dimensions and in more abstract spaces

We can view the idea of *continuity* in the infinite space of candidates as an extension of *stability* in the discrete case. With stability, a small change in a preference profile results in at most a relatively small choice in the output of the social choice rule.

In the Chichilnisky model, the possible alternatives (candidates) form a subset A of Euclidean n-dimensional space. She assumes each member of society has preferences over the set A, which vary smoothly as we move from one spot in A to another. She also assumed that preferences are *unsatiated*—that is, given any neighborhood U of a point x in A, there is some point y in U that you will prefer to x. We could then form a vector starting at x that points in the direction of greatest increase in our preference. For consistency purposes, we normalize the vector to have length 1.

Graciela Chichilnisky and Geoffrey Heal found a geometric description of when it is possible to solve problems like Dave and Judy's where we have k citizens instead of just two. They proved in 1983 that when each voter specifies an element from a space $\mathscr{B}$ of preferences, then a social choice rule that outputs an element of $\mathscr{B}$ for every k-tuple of elements of $\mathscr{B}$ can be constructed that is continuous, anonymous, and unanimous if and only if $\mathscr{B}$ is *contractible*. Although contractible is a technical term with a very precise meaning, you can think of it as meaning that there are no holes in $\mathscr{B}$ or that $\mathscr{B}$ does not surround a hole. Alternatively, a contractible space is one that can be continuously shrunk to a point inside itself.

In the Dave and Judy example with all points on the lakeshore under consideration, the social choice function assigns to each pair of angles $(\theta_{Judy}, \theta_{Dave})$ another angle, the output of the social choice rule. The set of all possible pairs of angles can be represented geometrically by a torus, the surface of a hollow doughnut. Since the preference space $\mathscr{B}$ in this case is a circle and the circle surrounds a hole, the circle is not contractible. Hence, there is no social choice rule in this situation that is unanimous, anonymous, and continuous.

Luc Lauwers (2000) calls the Chichilnisky-Heal result "The Resolution of the Social Choice Paradox" and observes that Chichilnisky's introduction of a topological approach to social theory "caused a major breakthrough in the disentanglement of the possibilities and limitations of preference aggregation. . . . Necessary and sufficient conditions to resolve the social choice paradox were established and new insights in the relationships between different aggregation axioms were obtained."

Are there reasonable situations in which the space of preferences would be contractible? One result suggests that if there is at least some "limited agreement" among the voters, then it is possible to have a continuous, unanimous, anonymous social choice rule. For example, if there is some fixed preference v on the circle no individual has, then the space of preferences is contractible.

Chichilnisky was successful in translating many of the important axioms of discrete social choice procedures into the continuous setting and discovering their geometric nature and how they related to certain classic results in topology. Her breakthroughs paved the way for many others to explore topological choice theory. Recently, Yuliy Baryshnikov (1993, 1997) discovered new, deeper connections between the discrete and continuous approaches to collective decision making. In his words, this work

> *demonstrates a remarkable interplay between two theories of social choice: a* topological *one, initiated and developed mainly by Chichilnisky, and the classical,* combinatorial *one, stemming from the work of Arrow. Both theories deal with the aggregation of preferences with apparently cardinally different notions of preferences. Recall that in the classical theory, the preferences are assumed to be given on discrete sets of alternatives and constitute a discrete set by themselves. This bounds the technique of the theory to be combinatorial. In the topological theory of social choice, the set of alternatives is assumed from the beginning to have the structure of a topological space . . .*
>
> *Until recently it has been implicitly assumed that the theories coexist but do not have much in common. The combinatorial setting was considered as primary and more natural and simple, while the topological one was usually seen as hi-tech fortresses with no life nearby. . . . Both theories are in fact much closer to each other than was commonly thought. Actually, I believe that they are in fact two different guises of the same theory which uniformly covers both discrete and continuous phenomena of the social choice theory.*

We don't yet have a complete unifying theory that satisfies everyone, but progress is being made. You can follow new developments in such journals as *Social Choice and Welfare*, *Theory and Decision*, *Economic Theory*, and *Voting Matters*. Perhaps you can make some discoveries yourself.

IX. Historical and Biographical Notes

A. Pliny the Younger

The question of what procedure to follow when a group of individuals must choose among more than two alternatives goes back in history at least as far as ancient Rome. In A.D. 105, Pliny the Younger recounts a decision facing the Roman Senate. The issue concerned the fate of the freedmen of the consul Afranius Dexter, who had recently died. The freedmen were former slaves whom Afranius had liberated and who were working as his paid servants. The senators considered three options: let these servants go free, banish them to a remote island, or execute them. (Roman practice was to execute the slaves immediately on the death of the master.)

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Pliny the Younger (61−113 A.D.), was a Roman lawyer, author, and natural philosopher. He witnessed and wrote about the eruption of Mount Vesuvius in August 79 during which his uncle and mentor Pliny the Elder died. His letters about Vesuvius were so keenly detailed that modern vulcanologists describe that type of eruption as Plinian.

Pliny was the presiding officer of the Senate. He favored leniency for the freedman, but he knew that those who supported his position, although they were the largest group numerically, did not command a majority. Thus, he called for a plurality vote, asking each of those who favored a particular outcome to go to a separate corner of the room. The head of the faction favoring execution quickly realized that the freedman would then be released to live as citizens of Rome. He persuaded his followers to drop their first choice and vote for banishment, which then commanded a majority.

B. Jean-Charles Borda

Jean-Charles, chevalier, de Borda (May 4, 1733−February 19, 1799), was a French mathematician, physicist, political scientist, and mariner. Born into a French aristocratic family, he spent much of his life as a naval officer and military engineer. At age 23, Borda wrote an important paper on projectile motion that led to his election as a member of the French Academy in 1764.

In 1770, he proposed the ranked preferential voting system now known as the *Borda count*. The French Academy used Borda's method for many years until Napoleon abolished it when he came to power in 1801.

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Jean-Charles de Borda

Borda fought with the French in the American Revolution, eventually commanding a fleet of six ships until the British captured him in 1782. Later in his career, he did considerable work on hydraulics and also helped define the meter as one ten-millionth of the distance from the North Pole to the Equator.

C. Marquis de Condorcet

Borda proposed his weighted voting as an alternative to the method advocated by the Marquis de Condorcet. Condorcet's most important work was an 1785 treatise *Essai Sur L'application de L'analyse À La Probabilité Des Décisions Rendues À La Pluralité Des Voix* (*Essay on the Application of Analysis to the Probability of Majority Decisions*). This 500-page work furthered the development of probability theory and laid out more completely a mathematical basis for social choice procedures. Condorcet proposed that the winning candidate should be the one who beats all other candidates in head-to-head elections. Such a candidate, as we have noted before, is called a Condorcet Winner. If, for example, there are four candidates *A*, *B*, *C*, and *D* and *A* defeats *B* in a simple majority runoff between the two, and similarly *A* defeats *C* and *A* defeats *D*, then *A* should be the winner. Borda agreed that if there was a Condorcet Winner, then it should be the group's choice, but thought it was impractical to insist on a Condorcet Winner, since there would be many ways that no candidate might be qualified.

Condorcet realized that there could exist situations where no such candidate exists and that intransitive results could result in such cases; we saw several examples in earlier sections of this chapter. He argued, however, that we should only consider social choice mechanisms that guarantee selecting the Condorcet Winner, if one exists. Because weighted voting does not guarantee the selection of such a candidate (see Exercise 66), Condorcet strongly opposed Borda's method.

Although Condorcet was responsible for seeing that Borda's paper, presented orally in 1770, was published at the same time his own *Essai* went to press, he could be quite contemptuous of his fellow Frenchman. Condorcet said Borda "likes nothing better than to waste his time drawing up prospectuses, examining machines, etc., and especially because, realizing he was eclipsed by other mathematicians, he abandoned mathematics for petty experiments. . . . Some of his papers display some talent, although nothing follows from them and nobody has ever spoken of them or ever will."

Condorcet's criticism of Borda was not an entirely an objective assessment. They had clashed earlier about what should be done with funds worth about $50,000 in today's dollars that the king owed the French Academy. Condorcet felt they should be used to pay his salary, whereas Borda felt they should be used to support experimental research.

Public domain

Marquis de Condorcet

Marie-Jean-Antoine-Nicolas de Caritat took his title Marquis de Condorcet from the town of Condorcet in Dauphiné, where his family resided. Born September 17, 1743, Condorcet was educated in Jesuit schools in Reims and then at the Collège de Navarre and the Collège Mazarin in Paris.

Condorcet displayed early talent in mathematics and was elected to the French Academy of Sciences at age 26. A 1772 volume on calculus he wrote was described by the eminent mathematician Lagrange as "filled with sublime and fruitful ideas."

Although he served in the administration of Louis XVI, Condorcet strongly supported the French Revolution. He advocated economic, religious toleration, abolition of slavery, free and equal public education, constitutionalism, and equal rights for women. After the Revolution, he served in the Legislative Assembly as a representative from Paris. Condorcet aligned himself with the moderate Girondists, who were ousted by the more radical Jacobins led by Robespierre.

Condorcet argued strongly against the new, hurriedly written, constitution that was drawn up by the Jacobins to replace the one that he himself had been chiefly responsible for drawing up. "Condorcet was no politician," wrote one of his biographers. "His uncompromising directness of manner and inability to suffer illogical windbags in silence made him many enemies and few friends." When a warrant for his arrest was issued, Condorcet went into hiding for half a year and eventually tried to flee from Paris but was caught and imprisoned on March 27, 1974. Two days later he was found dead in his prison cell. Whether he committed suicide, died from natural causes, or was murdered is still not known.

While he was in hiding, Condorcet wrote *Esquisse d'un tableau historique des progrès de l'esprit humain* (*Sketch for a Historical Picture of the Progress of the Human Mind*), now considered one of the major Enlightenment texts. Condorcet held a strong belief that human progress was linked to scientific discoveries and to mathematical and logical reasoning. He argued that there was an intimate connection between scientific advances and the spread of justice and human rights. His vision was that we could, through rational thought and the accumulation and sharing of knowledge, continually progress toward a utopian society.

D. Charles Dodgson (Lewis Carroll)

Most of the world knows him as Lewis Carroll, the author of the children's classic *Alice's Adventures in Wonderland*, its sequel *Through the Looking-Glass*, and the nonsense poems "The Hunting of the Snark" and "Jabberwocky." But Charles Lutwidge Dodgson (January 27, 1832–January 14, 1898) has an independent reputation as an Oxford University mathematician who made some serious contributions to social choice theory.

Public domain

Charles Lutwidge Dodgson (Lewis Carroll)

Table 6.27

Group	I	II	III	IV	V	VI	VIII
Size	2	2	2	2	2	1	1
1st choice	D	B	C	D	A	A	D
2nd choice	C	C	A	B	B	D	A
3rd choice	A	A	B	C	C	B	B
4th choice	B	D	D	A	D	C	C

In the mid-1870s, Dodgson wrote three short pamphlets about voting procedures to deal with multiple-candidate elections: *A Discussion of the Various Methods of Procedure in Conducting Elections* (1873), *Suggestions as to the Best Method of Taking Votes, Where More Than Two Issues Are to Be Voted On* (1874), and *A Method of Taking Votes on More Than Two Issues* (1876). Dodgson's practice as a mathematician was to develop his own solution to problems without considering previous work on the subject. Thus, it was unlikely that he had read the papers of Borda and Condorcet; he essentially came up with some of their approaches entirely on his own.

Dodgson suggested first looking for a candidate who was the first choice of a majority of voters. If such a candidate exists, then that person is declared the winner. If no one commands a majority, then Dodgson proposes examining all the two-candidate elections and seeing whether a Condorcet Winner emerges. If there is no Condorcet Winner, then Dodgson discusses various ways to proceed. These include versions of Instant Runoff Voting and the Borda count. He also proposed that the individual preference lists of the voters be examined to determine the smallest number of switches of consecutive candidates required to produce a Condorcet Winner. He envisioned a sequence of rounds of voting wherein the electors would be made aware of this information and offered the opportunity to submit new preference rankings.

Today, social choice theorists call an alternative a Dodgson Winner if it can be made a Condorcet Winner by interchanging as few adjacent alternatives in the individual rankings as possible. Consider, for example, the preference rankings of 12 voters among four candidates summarized in Table 6.27.

Candidate *D* has a plurality of five first-place votes, but no candidate commands a majority. Nor is there even a Condorcet Winner. If any three of the voters in Groups I–IV interchange their consecutive rankings of *C* and *A*, then *A* would become a Condorcet Winner. No other interchange by three or fewer voters of adjacent candidates in their rankings produces a Condorcet Winner. Hence, candidate *A* would be the Dodgson Winner.

E. Kenneth Joseph Arrow

The most prestigious and coveted international honors are the annual Nobel Memorial Prizes. These awards are given for outstanding achievements in medicine, literature, peace, chemistry, physics, and—since 1969—economics. The announcements of these awards each autumn are front-page news.

The Swedish Academy of Science selected Kenneth J. Arrow as a co-winner of the 1972 Nobel Memorial Prize in Economics. The academy cited Arrow's pioneering

contributions to general economic equilibrium theory and welfare theory. Although Arrow has made several key breakthroughs in economic theory, many of his colleagues rate the Impossibility Theorem of this chapter as his major achievement. According to the well-known economist Paul Samuelson, himself a Nobel Laureate in 1970, this theorem "is not only a stellar contribution to economics, it is as well a breakthrough for political science, and I would dare assert, for philosophy itself."

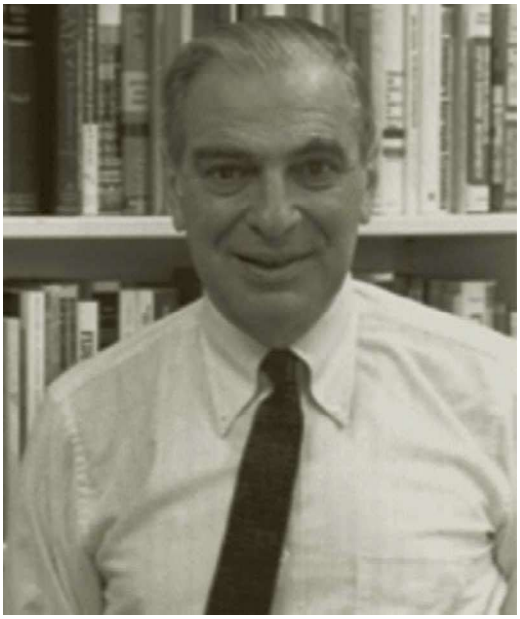

Photo courtesy of Kenneth Arrow

Kenneth J. Arrow

Arrow was born in New York City to Jewish immigrants raised on the Lower East Side. He graduated from the City College of New York in 1940 at the age of 18 with as he put it "a degree of Bachelor of Science in Social Science but a major in Mathematics, a paradoxical combination that was prognostic of my future interests." Arrow's advanced degrees were taken at another Manhattan institution, Columbia University. After a 4-year stint in the U.S. Army Air Force during World War II, Arrow was a research associate with the Cowles Commission at the University of Chicago from 1947 to 1949. The Impossibility Theorem was part of his Ph.D. thesis and in finished form was published as a book, *Social Choice and Individual Values*, in 1951.

Arrow began his work on the social welfare problem by trying to develop a reasonably fair function that took a collection of individual preference rankings and produced a group ranking. "I just started playing around," he told one interviewer. "It took me about two days to decide I was on the wrong track because I was looking for some solution. It didn't occur to me that there was no solution."

In 1949, Arrow joined the faculty of Stanford University where he taught for almost 20 years and was a major force in developing at Stanford an outstanding group of economic theorists and mathematical model builders. He also worked briefly with the Council of Economic Advisers during the administration of President John F. Kennedy. In 1968 Arrow moved to Harvard University where he became the James Bryant Conant University

Professor in 1974. In 1979, he returned to Stanford University with the position of Joan Kenney Professor of Economics and Professor of Operations Research. Arrow formally retired in 1991 but continues to be an active participant in economics conferences and was a vigorous bicyclist well into his mid-eighties.

Arrow has written or edited many books and dozens of papers whose topics include the mathematical theory of inventory and production, time series analysis of interindustry demands, linear and nonlinear programming, public investment and optimal fiscal policy, the theory of risk bearing, and general competitive analysis.

"Despite the deep abstraction of his econometric theories, friends consider Professor Arrow basically a humanist, a scholar who has always tried to apply fundamental theory to such social problems as medical care, education, race discrimination and water resources" wrote Robert Reinhold [1972] in a *New York Times* profile.

In appraising the work for which Arrow received a Nobel Prize, Samuelson [1972] wrote, "Men have always sought ideal democracy—the perfect voting system. . . . What Kenneth Arrow proved once and for all is that there cannot possibly be found such an ideal voting scheme. The search of the great minds of recorded history for the perfect democracy, it turns out, is the search for a chimera, for a logical self-contradiction. . . . Aristotle must be turning over in his grave. The theory of democracy can never be the same . . . since Arrow."

Despite his many honors and the demands of his research, Arrow has been remarkably available to undergraduate students and younger colleagues. He was the only senior faculty member at Harvard, for example, who volunteered to take on an assignment to lead discussion sections of an introductory economics course, a task usually delegated to graduate student teaching assistants. Four of Arrow's graduate students have themselves been awarded the Nobel Prize in Economics.

President George W. Bush presented the National Medal of Science, the nation's highest scientific honor, to Kenneth Arrow in November 2005 for "groundbreaking contributions to the pure theory of economics but [he] also holds a broad understanding of the social science arena in which theories are confronted and practical lessons worked out. His fundamental research on risk perception and behavior under uncertainty, and on equilibrium in markets with imperfect information, began a revolution in the design and analysis of market allocation mechanisms."

Photo by Eric Draper, Courtesy of the George W. Bush Presidential Library & Museum

President George W. Bush about to present the National Medal of Science to Kenneth Arrow

F. Amartya Sen

Amartya Kumar Sen, discoverer of the Liberal Paradox, is a Bengali economist and philosopher born on November 3, 1933. He won the Nobel Memorial Prize in Economic Sciences in 1998 "for his contributions to welfare economics" for his work on famine, human development theory, welfare economics, the underlying mechanisms of poverty, and political liberalism.

From 1998 to 2004, Sen was Master of Trinity College at Cambridge University, becoming the first Asian academic to head an Oxbridge college.

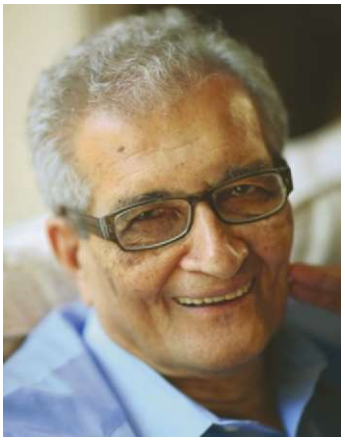

Photo by Graeme Robertson. Used with permission of SIPA USA

Amartya Sen

Among his many contributions to development economics, Sen has produced work on gender inequality. He is currently the Lamont University Professor at Harvard University. Sen's books have been translated into more than thirty languages.

G. Graciela Chichilnisky

One of the world's preeminent mathematical economists, Graciela Chichilnisky has achieved great success despite numerous obstacles. She is currently UNESCO professor of economics and mathematics and professor of statistics at Columbia University, where she also directs the Program on Information and Resources and the Center for Risk Management.

Chichilnisky was born March 27, 1946, in Buenos Aires, Argentina. Her parents Salomon Chichilnisky and Raquel Gavensky came from families fleeing Russian anti-Semitic pogroms at the turn of the 20th century. Her father battled his way from being a dock worker to becoming a medical doctor, then a professor of neurology, and later the national Secretary of Health, in which role he built many hospitals and a large part of Argentinean National Health system.

Chichilnisky excelled at school despite several anti-Semitic incidents. As a high school junior, she informally took courses at the University of Buenos Aires, where she gravitated to mathematics from an initial focus on philosophy and sociology.

> *"I wanted to do mathematics that would be applied to resolve social problems," she recalls. "I thought that studying Mathematics first would give me a control of the 'technology' that economists use to validate their theories and their policies. I felt it was important to 'control the*

technology'—rather than 'be controlled by it'—since many economists appear to fear the mathematical foundations of economics and adopt theories or policies based on what they learn from others mathematical models. I always liked the idea of creating my own mathematical models, rather than adopting somebody else's. Mathematics was a pleasure to learn. I think of it as the natural language that the brain uses to communicate with itself."

Just as Chichilnisky was about to start college, a military junta took over Argentina and closed down the universities. Fortunately she was invited to begin graduate level studies in mathematics at the Massachusetts Institute of Technology. She arrived in Cambridge, a single mother with infant child, without much knowledge of English, competing with doctoral students at a leading university without benefit of an undergraduate degree. Soon, however, she was at the top of her class with scholarship support from the Ford Foundation. She transferred to the University of California at Berkeley when she completed her Ph.D. in mathematics with her thesis "Group Actions on Spin Manifolds," a topic at the intersection of algebraic topology and physics.

Chichilnisky's first major job was as Director of Modeling at Fundacion Bariloche in Argentina, where she created a mathematical model of the world economy with an interdisciplinary team of prominent Latin American scientists, including geologists, sociologists, population experts, computer scientists, political scientists, and economists. In this model, she created the concept of "development based on the satisfaction of basic needs" rather than a more traditional approach of development through maximizing Gross Domestic Product. A radical idea at the time, Basic Needs was adopted as the cornerstone of efforts to define sustainable development by 166 nations at the Earth Summit in 1987.

Seeking a better understanding of international markets, Chichilnisky decided to return to Berkeley to work on a second doctoral degree, this time in economics. Her primary advisor was another Nobel Laureate, Gerard Debreu. While completing her dissertation, she also worked as a research associate at Harvard with Kenneth Arrow, who became an important mentor for her. After a period teaching mathematics and economics at Harvard while beginning her breakthrough research on topological choice theory, Chichilnisky moved to New York in 1977. There she served as director of research for the United Nations Institute for Training and Research while teaching at Columbia and Harvard.

Chichilnisky has published more than 16 books and 300 research papers covering a broad range of concerns, including oil in the international economy, development and global finance, information and uncertainty in markets, equity and efficiency in environmental markets, the gender gap, sustainability, dynamics, and uncertainty. She has received many honors and awards during her career. The Greek government named her Global Citizen of the Year in 2007, the University of Oslo conferred the Leif Johansen award on her in 1995, and *Hispanic Business* listed her as one of the most influential Latinos in the United States in 2006. She has frequently been mentioned as a possible future winner of the Economics Nobel Prize.

Chichilnisky has worked extensively on the Kyoto Protocol process, creating and designing the concept of the carbon market that has become international law in 2005. Working closely for several years with negotiators of the United Nations Framework Convention on Climate Change, the organization in charge of deciding world policy with respect to global warming, Professor Chichilnisky acted as a lead author of the

Intergovernmental Panel on Climate Change. The IPCC received the 2007 Nobel Prize for their work in this area.

Photo reproduced by permission of Graciela Chichilnisky

Graciela Chichilnisky

The road to such achievements has often been a very bumpy one for Chichilnisky. "Somewhere along this path," she writes, "I met uncontrollable forces, full of sound and fury, that thrust me up close into the stormy transition of women's roles at the turn of the 21st century." Her mentor at MIT suggested that she would not be able to compete successfully with students in the program there and suggested she transfer to a less demanding discipline at a weaker university. Later there were false malicious rumors that each of her doctoral theses were not original works, but had been written by her adviser or other senior colleagues. "As a foreign woman and a single mother, I started to face bewildering circumstances," Chichilnisky writes:

> *I learned a great deal during this process, about how women's intellectual property is treated in the masculine world of academia. . . . All this made me aware in the years that followed of the plights of other women in Ivy League universities whose intellectual work had been stolen or duplicated with impunity, or attributed to others. . . . While it is hard enough to compete with men in academic research, obtaining credit for what one has accomplished proved to be much more difficult . . . academic citations are consistently biased against women. Men resist*

giving women the basis for measuring academic achievement. They successfully deny them credit for their work in the form of academic citations. This problem is true today and in my case it has only become worse as my work has become better known.

Hostility escalated . . . as I grew professionally, it became relentless. Some of my colleagues recommended to my students not to work with me, others wrote threatening letters to my sources of funding and published numerous articles against my work; others acting as editors limited the ability of authors to write extending my work in this area. The campaign against my work extended to the Columbia administration, and created a hostile climate in which it was very difficult to work. . . .

At Columbia I have excellent colleagues who tried to stop this trend, but the forces of darkness succeeded. . . . Eventually my assistants' offices at the Economics Department were destroyed and my own office made unusable, my courses were removed, I was marginalized and treated with hostility, and my salary became so low that years later it had to be almost doubled and still remained below the average of male full professors.

When Chichilnisky discovered that her salary was substantially lower than her colleagues, she sought to remedy the discrepancy in what became "a David and Goliath epic." She describes the administration's response as one of "indifference and scorn." After failing to resolve the issue internally, Chichilnisky filed a lawsuit against Columbia in 1990. A settlement was reached in 1995 awarding her $500,000 and commitments from the university to provide the Program on Information and Resources, which Chichilnisky directed, with space and other support. By 2000, she came to believe that Columbia was not honoring its promises, was harassing her in retaliation for challenging the university, and was continuing to pay her unfairly. She filed suit again. Columbia filed counterclaims alleging she was delinquent in her duties and had secretly operated a private consulting company. After nearly 8 years of legal wrangling, the parties reached anew settlement in June 2008 awarding Chichilnisky $200,000. "The exact number isn't as important as the principle that it was a substantial amount of money that the university had to pay," she said. "And that has to do with who is right and who is wrong."

Despite the obstacles and hostility Chichilnisky has had to face, she remains positive:

For a woman to survive and to thrive she must learn to turn negative responses into positive resources. This is a perverse reversal to the Pavlovian response. I call this, for short, "turning dung into fertilizer." I believe it is one of the most important elements for women's success and happiness. It is a wonderful recipe for dealing with the "glass ceiling", a well-known and somewhat cruel situation where the more you succeed, the more you get punished. Think of it this way—energy is energy—and simply changing the sign of the response one receives from negative to positive allows one to use all the energy received constructively, and turn it into a survival tool. In mathematical terms, this is "life modulo two." It is the absolute value of the response that counts, not the sign . . . the only genuine source of happiness in life is the feeling of being useful to others. Nothing else does the job. This is true for anybody. It is not achievement or success, it is not money. It is this feeling of being useful that counts.

EXERCISES

I. Three Voting Situations

1. Show that a voting mechanism that gives a satisfactory resolution of situations in which a single best alternative must be chosen (as in Example 1) can be modified to handle situations when a full ranking of various alternatives is required (as in Example 2).

2. Are there any voting situations essentially different from those described in Examples 1–3? How are the outcomes determined in such situations?

II. Two Voting Mechanisms

3. What safeguards protect minority rights in systems using simple majority voting?

4. Suppose the senators are split in the following manner:
(A B C) 49 votes
(B C A) 49 votes
(C A B) 2 votes
If modified simple majority voting is used here, will the judgments of the Senate be transitive?

5. If the senators are split in the following manner:
(A B C) 32 votes
(B C A) 33 votes
(C A B) 35 votes,
show that there is a Condorcet Winner.

6. If 51 senators share the preference ranking (A B C), show that the Senate will have transitive preferences. Is there a Condorcet Winner?

7. Is it necessary for a majority of senators to share a common preference ranking to guarantee that the Senate judgments will be transitive? Why?

8. Consider a legislative body that only passes resolutions if they are supported by more than two-thirds of the members. How would such a body settle questions like those proposed by Examples 2 and 3? What inequities does such a system possess?

9. In order to correct past discrimination, it has been proposed that for a limited period, the votes of women be given twice the consideration of the votes of men—that is, each woman receives two votes on every proposal while each man receives one. Proposals are adopted or rejected on a simple majority count. Can intransitive results emerge? What other injustices are associated with such a system?

10. There are 1,000,000 shares of stock in the Emerson Construction Company. Two shares are owned by Mrs. Emerson, and the remaining shares are split evenly between her two sons. In deciding company policies, each shareholder has a number of votes equal to the number of shares he controls. How much relative power does Mrs. Emerson have?

11. The outcome of a weighted voting mechanism depends not only on the rankings of the individual judges, but also on the points assigned to each place in the rankings. For the beauty contest example described in the text, determine the rankings of the contestants if a second place is worth only 3 points.

12. In some gymnastics competitions, four judges individually assign a number between 1 and 10 to each contestant. The highest and lowest scores are discarded and the contestant receives the average of the two intermediate scores. What injustices would be associated with such a voting mechanism?

13. In many voting situations, the individual voter is permitted to designate more than one contestant as his preference, but is not allowed to rank-order his preferences. For example, 10 candidates may be running for three positions on a local school board. Each voter may place X's besides the names of three candidates. The candidates who receive the largest number of X's are the winners. How fair is such a voting mechanism?

14. Under *plurality* voting, the candidate with the largest number of voters, even if it is not a majority, is declared the winner. Show that with the hypothetical distribution of senators' preferences on tax policy we studied, alternative C would be the plurality winner. Are there legitimate objections to that outcome? Imagine, for example, the 65% of the senators who agreed that B was a better choice than C.

15. *Instant Runoff Voting* (*IRV*) is a procedure that has been gaining substantial support in recent years. If no candidate receives a majority of first-place votes, then the candidate with the fewest first-place votes is eliminated and that candidate's voters are then assigned to the person ranked second on the ballots that named that candidate as first choice. Show that under IRV and hypothetical distribution of senators' preferences on tax policy, alternative A would be eliminated and the 31 votes it received would all be

transferred to alternative B. What inequities does IRV suffer from?

16. Under *Traditional Runoff Voting* (*TRV*), the two candidates with the highest numbers of first-place votes advance to a second election between the pair.

(a) Show that TRV and IRV produce the same result if there are exactly three candidates.

(b) Construct, if possible, a set of rankings among four candidates that yields different winners under TRV and IRV.

III. An Axiomatic Approach

17. Show that simple majority voting satisfies Axioms 1–5 if there are exactly two alternatives.

18. Which of the axioms are satisfied by weighted voting mechanisms?

19. Construct voting mechanisms that satisfy all the axioms except
(a) Axiom 1, (b) Axiom 2, (c) Axiom 3, (d) Axiom 4, (e) Axiom 5

20. Weaken Axiom 2 by eliminating transitivity of societal preferences, and construct various mechanisms that satisfy the new set of axioms.

21. Weaken Axiom 2 by eliminating the demand for a unique societal preference, and construct a mechanism that satisfies the new set of axioms.

22. In what way does Axiom 4 eliminate the possibility of voters manipulating the system by disguising their true preferences?

IV. Arrow's Theorem

23. Construct a voting mechanism for which there is a set V of voters and a pair of alternatives x and y so that V is decisive for x against y, but V is not decisive for y against x. Can you construct such a mechanism that satisfies all but one of Arrow's axioms?

24. The eight-person society discussed under remark (d) has a mechanism that does not satisfy the Unanimity Axiom. Why? Suppose the mechanism is modified so that for every pair x, y of alternatives, x is socially preferred to y whenever everyone prefers x to y; otherwise, the societal preference is the opposite of Mike's. What axioms does this system satisfy?

25. **(a)** Prove that a minimal decisive set will always exist if there is a finite number of voters provided Axiom 2 is satisfied.

(b) Construct a voting mechanism for a society with an infinite number of voters in which minimal decisive sets do not exist—that is, show that if V is any set of voters decisive for some pair of alternatives, then there is a proper subset of V that is also decisive for some pair of alternatives. Can you construct such a mechanism that satisfies all of Arrow's axioms?

26. What happens in the proof of Claim I if the minimal decisive set is the set of all voters? Can this happen in a system satisfying Axioms 1–4?

27. In the proof of Claim I, it is tacitly assumed that V' is non-empty. Can you prove that V' always contains at least one voter?

28. **(a)** Prove that $a\overline{D}b$ implies aDb.

(b) Find an example in which aDb is true, but $a\overline{D}b$ is not.

29. Prove that xDy implies $z\overline{D}x$ and $y\overline{D}x$ if there are exactly three alternatives x, y, and z—that is, verify (5) and (6) of Case 1.

30. Verify the details of the argument of Case 2.

V. Theorem of Gloomy Alternatives

31. Our proof of the Liberal Paradox Theorem assumed that all four of the alternatives w, x, y, and z were distinct from one another. Prove the theorem if this is not true.

32. Some authors reserve the term *Social Decision Function* (*SDF*) for mechanisms that assign to each possible profile of preferences a single top choice or winner rather than a full societal ranking. Suppose we have precisely two voters and two candidates and each voter rank-orders the candidates.

(a) Show that there are four possible profiles and, since there are two possible winners for each profile, show that there are $2^4 = 16$ possible SDFs.

(b) How many possible SDFs are there if there are three voters and two candidates?

33. If there are two voters and three candidates, show there are 36 possible profiles and hence 3^{36} different possible SDFs.

If there are n voters and k candidates, how many different possible SDFs are there?

34. A social decision function is *nondegenerate* or *nontrivial* if for each possible candidate there exists at least one profile of voter preferences under which that candidate is chosen as the winner.

(a) Which of the 16 possible SDFs of Exercise 32 are nondegenerate?

(b) For a situation with exactly two voters and three candidates, exhibit an SDF that is trivial and an SDF that is nondegenerate.

35. Prove the following proposition: Suppose we have a society with two voters and we need to choose a winner among exactly three alternatives using an SDF that is nondegenerate and strategy-proof. If both voters prefer candidate A to candidate B, then B cannot be the winner using this SDF.

36. Use the proposition of Exercise 35 to prove the Gibbard-Satterthwaite Theorem in the case of a two-voter, three-candidate situation. In particular, show that any nondegenerate strategy-proof SDF that returns a winner for every profile of rankings must be dictatorial.

VI. Instant Runoff Voting

37. Table 6.28 below shows some results for the 1990 presidential election in Ireland. The three candidates were Mary Robinson (*MR*) of the Labour Party, Brian Lenihan (*BL*) of the traditionally dominant Fianna Fail Party, and Austin Currie (*AC*) of the Fine Gael Party. A total of 1,584,095 people voted in the election; of these, 9,444 indicated no preference for president. Note that the voters in Group I only listed Robinson as first choice; they did not indicate a second. The voters in Group II listed Robinson first and Currie second, but showed no third choice explicitly.

TABLE 6.28

GROUP	I	II	III	IV	V	VI	VII	VIII	IX
SIZE	306133	183679	122453	208345	347242	138897	205565	46789	25548
1st	*MR*	*MR*	*MR*	*BL*	*BL*	*BL*	*AC*	*AC*	*AC*
2nd		*AC*	*AC*		*AC*	*AC*	*MR*	*BL*	
3rd			*BL*			*MR*			

(a) Who was the plurality winner?

(b) Who is the winner under Instant Runoff Voting?

38. In what sense is the winner of an IRV procedure dependent on which candidate is eliminated first?

39. There are at least two ways that IRV can be modified to produce a societal preference ranking among all the candidates after determining the first-place finisher: (I) Give second place to the last candidate eliminated, third place to the next to last eliminated, and so forth, or (II) Go back to the original individual rankings, cross out the winner, and apply IRV to the result; the winner of this election is the second-place finisher. Continue similarly to find third place, fourth place, and so on.

(a) Apply (I) and (II) to the example shown in Table 6.11.

(b) Apply (I) and (II) to the example in Exercise 37.

(c) Do (I) and (II) always produce the same societal ranking?

(d) Suppose you use method I. Which of Arrow's Axioms are satisfied, and which are violated?

(e) Suppose you use method II. Which of Arrow's Axioms are satisfied, and which are violated?

40. Verify that in the Fishburn-Brams example with candidates Bitt, Huff, and Wogg (Table 6.21), Mr. Huff wins each possible two-person race.

41. For the Fishburn-Brams example with the Smiths voting, show that if two or more of the 82 voters in Group II had switched the order of Bitt and Wogg so that Wogg became their top choice, then Huff would become the winner, again demonstrating that increasing support for a candidate can lead to his defeat!

42. Show that if the Smiths had voted in any order that did not have Mrs. Bitt in first place, then Mr. Huff would have won.

43. Suppose the Smiths not only voted, but also recruited 300 other additional people who shared the Smiths'

preferences but were not intending on voting to show up at the polls and submit their rankings. Who wins the election under IRV?

44. Here is another interesting problem that Fishburn and Brams found can arise with IRV. Suppose the town of Branburn is divided into two districts, East and West, with the numbers of voters in each district with the possible preference rankings summarized in this table:

Group	I	II	III	IV	V	VI
Total	417	82	143	357	285	324
East	160	0	143	0	0	285
West	257	82	0	357	285	39
1st	Bitt	Bitt	Huff	Huff	Wogg	Wogg
2nd	Huff	Wogg	Bitt	Wogg	Bitt	Huff
3rd	Wogg	Huff	Wogg	Bitt	Huff	Bitt

Show that we apply the IRV process to each district separately, then Bitt wins in the East and she wins in the West, but she does not win if the entire town is considered as one district. Fishburn and Brams call this an example of the *Multiple Districts Paradox*: A candidate can win in each district separately, yet lose the general election in the combined districts.

45. Suppose there are nine voters with the following preference rankings for candidates A, B, and C:

Group	I	II	III
Size	4	3	2
1st choice	*A*	*B*	*C*
2nd choice	*C*	*C*	*A*
3rd choice	*B*	*A*	*B*

Show that under plurality or instant runoff voting, A wins the election even though C is a Condorcet candidate—that is, a majority prefers C over A and a majority prefers C over B.

46. Prove that under plurality or instant runoff voting with precisely 3 candidates, a Condorcet candidate will always lose if that candidate is not one of the top two candidates listed in first place. Is it true that a Condorcet candidate will always win if that person is one of the top two? Construct an example, if possible, of a 4 candidate contest with candidates A, B, C, and D where C is a Condorcet candidate, C wins under instant runoff voting, but A and B each receive more first-place votes than C.

VII. Approval Voting

47. Suppose we have a population of 120 people ranking three alternatives A, B, and C. We can divide the population into five groups whose numbers and orderings of the alternatives are shown in the table below:

Group	I	II	III	IV	V
Size	36	12	26	20	26
1st choice	*A*	*B*	*B*	*C*	*C*
2nd choice	*B*	*C*	*A*	*B*	*A*
3rd choice	*C*	*A*	*C*	*A*	*B*

(a) Show that C is the plurality winner.

(b) Show that A is the Condorcet Winner.

(c) Show that if first-place votes earn 3 points, second-place votes 2 points, and third-place votes 1 point, then the Borda count winner is B.

(d) If we use Approval Voting and each person votes for her top two choices, who wins?

(e) Suppose we use Approval Voting and half of each group votes for their top two choices and the other half only votes for their top choice, who wins?

(f) Who is the winner under Instant Runoff Voting?

48. One argument for Instant Runoff Voting (IRV) over Approval Voting (AV) goes as follows:

Approval voting has another important real-world flaw. Political behavior has much to do with what is rewarded by the election system, and AV would exacerbate one of the worst aspects of U.S. campaigns: avoidance of substantive policy debate. Because a candidate could lose despite being the first choice of an absolute majority of the electorate, smart candidates would avoid controversial issues that alienate any significant number of voters. Smiling more and using policy-empty themes like 'I care' will not clarify the important choices leaders must make. Those rewarded by AV could be characterized as 'inoffensive' more than 'centrist.' IRV strikes a better balance. It rewards

candidates who stand out on policy enough to gain first-choice support, yet encourages coalition-building and fewer personal attacks as candidates seek to be the second choice of other candidates' supporters. [http://archive.fairvote.org/op_eds/science2001.htm]

Discuss the merits of this argument. Could a "smart" candidate in an IRV situation also do well by being "likable" by being "inoffensive" and garner a lot of second choice?

49. Those who favor Approval Voting over Instant Runoff Voting sometimes claim that while it may be "rational" under IRV for a voter to list her second choice as her top choice in the submitted preference ranking, it is never the case under AV that you should vote only for your second choice; you should vote for only your first choice or for your top two choices. Is this claim valid?

50. For our original Senate example with 31 voters having the preference (ABC), 34 having (BCA), and 35 having (CAB), find the winner under Approval Voting if all voters list their top two choices as approved.

51. A voting system is *determinate* if each profile of voters' preference rankings uniquely determines a group preference ranking. One deficiency of Approval Voting is that it is not determinate: given a complete list of everyone's preference rankings, there may be many possible outcomes depending on exactly how many voters approve of more candidates than their first choice. As an example, suppose we have 15 voters whose rankings among candidates A, B, and C divide into three groups with different numbers as shown in this table:

Group	I	II	III
Size	6	5	4
1st choice	*A*	*B*	*C*
2nd choice	*C*	*C*	*B*
3rd choice	*B*	*A*	*A*

(a) Show that if all voters only check their top choice, then candidate A wins.

(b) Show that if exactly two voters of Group III check their top two choices and everyone else only checks their first choice, then candidate B wins.

(c) Show that if exactly three members of Group I check their top two choices and everyone else only checks their first choice, then candidate C wins.

(d) What is the result if four members of Group I and three members of Group III check their top two choices?

(e) What is the result if two members of Group I and one member of Group III check their top two choices?

VIII. Topological Choice Theory

52. Dave and Judy are deciding between two possible locations A and B. They agree on a decision rule that will select location A unless both of them name B as their top choice, in which case B is declared the winner. Does this rule satisfy the properties of anonymity, unanimity, and stability?

53. Suppose Dave and Judy are trying to decide among three possible locations A, B, C along the lakeshore. A proposed decision rule is to choose the common site if they both agree, but to pick the unnamed site if they disagree. Thus, if Dave submits A and Judy submits C, the decision rule outputs B. Show that this rule satisfies the properties of anonymity, unanimity, and stability.

54. Show that the assignment shown in the matrix below violates the stability criterion.

		DAVE'S A	TOP B	CHOICE C	D
JUDY'S	A	*A*	*A*	*B*	*D*
TOP	B	*A*	*B*	*C*	*D*
CHOICE	C	*B*	*C*	*C*	*C*
	D	*D*	*D*	*C*	*D*

55. For the David and Judy problem with five possible home sites, find social choices that are:

(i) Unanimous and anonymous but not stable

(ii) Unanimous and stable but not anonymous

(iii) Stable and anonymous, but not unanimous

56. A *metric* on a set S is a real-valued function d that assigns a nonnegative number $d(x, y)$ to every pair of elements x and y of S in such a manner that for all x, y, and z in S, the following are true:

(i) $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

(a) If S is the set of real numbers, show that the function $d(x, y) = |x - y|$ is a metric.

(b) If S is the set of points in the plane, show that the function $d(x, y) =$ length of line segment between x and y is a metric. Note that if $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then $d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ is a metric.

(c) If S is the set of all n-dimensional vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ where each x_i is a real number, show that $d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ is a metric. This metric is called the *Euclidean metric* or Euclidean distance function.

(d) If $n = 1$, how do the metrics in parts (c) and (a) compare?

57. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any two points in the plane. Show that the function d defined by $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$ is a metric. Can you say why this function d is called the *taxicab* metric?

58. Show that the ε–δ definition of continuity is equivalent to the *neighborhood* definition.

59. **(a)** The *norm* of a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is the real number $||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$. Show that the norm of a vector is the Euclidean distance between the origin $0 = (0, 0, \ldots, 0)$ and the vector.

(b) If $\mathbf{v}$ is a nonzero vector, then show that $\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||}$ is a vector of length 1 pointing in the same direction as $\mathbf{v}$.

60. Show that there is no social choice rule for the David and Judy problem with five possible home sites that is unanimous, anonymous, and stable.

61. Let X be the set of points on the unit circle except for the point $(-1, 0)$. Show that each point P on X can be described uniquely by an angle θ where $-\pi < \theta < \pi$. Then consider the function F that sends the pair (θ, t) to the point $(1 - t)q$ for $0 \le t \le 1$ Show that F sends each point of the form $(\theta, 0)$ to itself and each point of the form $(\theta, 1)$ to the point with Cartesian coordinates $(1, 0)$. Finally, show that F is a continuous function of θ and t. Thus the effect of F is to continuously shrink X to a point in X, always staying inside X. Such a function is called a *retraction*. A space is *contractible* if there is a retraction of X onto a single point in X. Thus, the circle minus a point is contractible.

62. Suppose our city council initially wishes to divide its expenditures into two categories: Municipal Services (M) and Education (E). The council will spend at least \$10 million, but no more than \$20 million.

(a) Sketch the set of all possible choices.

(b) Sketch the set of all possible choices if each of M and E must be at least \$5 million.

63. Are the regions in Exercise 62 (a) and (b) contractible?

64. The city council needs to pick a site for a new water treatment plant. It decides that the plant must be at least 2 miles from the center of the city, but no more than 3 miles from the center. Sketch the set of all possible locations. Is this set contractible?

65. Suppose the preference rankings among nine voters are distributed as in Exercise 45.

(a) Show that if 10 points are awarded for each first-place ranking, 5 points for second, and 2 points for third, then A wins a Borda count even though C is the Condorcet Winner.

(b) If N is any positive integer and N points are given for first-place rankings, $N - 1$ for second and $N - 2$ for third, then C wins under a Borda count.

66. Construct an example of preference rankings where a Borda count with N points for first place, $N - 1$ for second, $N - 2$ for third, and so forth yields a winner who is not the Condorcet Winner.

67. Consider the preference rankings given in Table 6.27.

(a) Show that if any three voters belonging to Groups I–IV interchange their consecutive rankings of C and A, then A becomes a Condorcet Winner.

(b) What is the smallest number of interchanges of consecutive candidates needed to make C the Condorcet Winner? Answer the same question for candidates B and D.

SUGGESTED PROJECTS

1. Instead of discarding some axioms or weakening them to obtain a consistent set, you might think about strengthening the first axiom. Axiom 1 allows voters to list the alternatives in order of preference, but does not allow for expression of intensity of differences between alternatives. Two voters may both list x and y

at the top of their lists, although the first voter's feelings are almost indifferent toward the two, while the second voter much prefers x to y. Investigate methods of incorporating intensities into individual preference lists. Discuss the consistency of sets of axioms allowing for such measures.

One such approach is *Range Voting*, a system whereby each voter assigns a measure on a scale of 0 to 100 to each candidate where the score reflects the strengths of the voter's preference for that candidate or voter's rating of the candidate's worth or appeal to the voter. The scores are added; the candidate with the highest total score is the winner. Range Voting can be interpreted as a generalization of approval voting that uses a scale of 0 (disapprove) to 1 (approve). Show that we can derive each voter's preference ranking of the candidates from the scores given out. What are the strengths and weaknesses of Range Voting?

2. Peter Fishburn has shown that the Axioms 1−5 are consistent if there are an *infinite* number of voters. Investigate his proof. What is the real-world relevance of this result?

3. Some voting theorists have argued that the modified simple majority vote system is satisfactory because intransitivity rarely occurs. Is there some way of measuring the likelihood of intransitivity? Can you find instances in Senate voting where proponents have used intransitivity to their advantage by adjusting the agenda?

4. Can the standard voting systems (simple majority, weighted voting, and so on) be characterized axiomatically? H. P. Young isolated three characteristics of voting systems, which he termed "consistency," "the cancellation property," and "faithfulness." He was able to prove that any mechanism that is consistent, faithful, and has the cancellation property must be a weighted voting system. Are these three properties reasonable ones? Check the details of Young's proof. Derive, if possible, an axiomatic characterization of simple majority voting.

5. Develop a proof of Arrow's Theorem showing that Axioms 1, 2, 4, and 5 imply that Unanimity is violated.

6. Show how the proof for the Muller-Satterthwaite Theorem can be easily modified to prove Arrow's Theorem.

7. A *Condorcet method* is any election method that always selects the Condorcet Winner, the candidate who would beat each of the other candidates in a run-off election, if such a candidate exists. Which of the social choice rules we have studied are Condorcet methods? Arthur Copeland suggested one of the simplest Condorcet methods in 1951: pick the candidate who beats the largest number of other candidates in run-off elections. Investigate the properties of the Copeland method.

8. Perhaps the most current widely used Condorcet method is a process suggested by Markus Schulze in 1997. Although the Schulze method (also known as *Path Voting*) fails to satisfy the Independence of Irrelevant Alternatives Axiom, it does satisfy a number of other desirable criteria for a fair voting mechanism. Determine how the Schulze method works, what societal rankings it outputs for the examples of profiles we have see in this chapter, and proofs that it satisfies some of the fairness criteria and fails others. Begin with Schulze's paper "A New Monotonic and Clone-Independent Single-Winner Election Method," *Voting Matters* **17** (2003): 9−19. Schulze expands on this work in a number of papers available on his website http://m-schulze.webhop.net

9. *Computational Social Choice* is a new field that explores questions at the interface of social choice theory and computer science. One of its main concerns is the efficiency of algorithms to implement the variously suggested voting mechanisms. How rapidly, for example, does the number of computational steps increase for a particular mechanism as the number of individual voters or individual candidates increase? Which mechanisms that appear attractive in theory may be infeasible to implement in practice because it would take the fastest computers hundreds of years to determine the winner? Are there social decision procedures that in theory are not strategy-proof but that in practice pose a computationally intractable problem to determine how to manipulate them by insincere rankings? Investigate what is known about the efficiency of particular social choice procedures. See Chevaleyre et al. (2005) for a good introduction to this interdisciplinary field.

10. Manipulable voting processes pose a serious problem for many social choice theorists as they may encourage voters to disguise their true preferences when submitting their rankings of the candidates. Others see that manipulation can take different forms and some may be a healthy aspect of democracy. Keith Dowding and Martin Van Hees distinguish between *sincere* and *insincere* manipulation and examine a class of social choice functions that are immune to one or the other

form. Investigate the claims they make in their provocatively titled paper "In Praise of Manipulation," *British Journal of Political Science* **38** (2007): 1–15 and examine what they imply about the main voting processes we have discussed.

11. Although Instant Runoff Voting and Approval Voting appear now to be the leading candidates to replace current methods of collective decision making, there are also strong arguments for using the Borda count. Examine Donald Saari's (2001) arguments on this option.

12. The Coombs Rule is an interesting variant of Instant Runoff Voting. This rule, suggested by mathematical psychologist Clyde Coombs, eliminates the candidate with the most last-place votes rather than the one with the fewest first-place votes. Investigate the properties of the Coombs Rule. In what ways is it superior to IRV? A good starting reference is Bernard Grofman and Scott L. Feld, "If You Like the Alternative Vote (a.k.a the Instant Runoff), Then You Ought to Know about the Coombs Rule," *Electoral Studies* **23** (2004): 641–659.

13. Many of the issues, definitions and techniques that arise in social choice theory are related to issues of *Fair Division* and *Apportionment*. Fair Division, also known as the *cake-cutting problem*, is the problem of dividing a resource in such a way that all recipients believe that they have received a fair amount. An important apportionment problem is allocating the 435 seats in the U.S. House of Representatives among the states in proportion to their populations with the constraints that each state gets a whole number of representatives. Examine the various methods proposed to solve such problems. A good starting reference is H. Peyton Young, *Equity in Theory and Practice*, Princeton: University Press, 1994.

14. Students with a background in algebraic topology would enjoy preparing an exposition of some of the results of Chichilnisky, Heal, and Baryshnikov listed below in the References.

15. Investigate social decision procedures that combine preference rankings and approval voting. In such systems, a voter either (a) submits a ranking of all the candidates, drawing a line between those approved by the voter and those not approved, or (b) submits a ranking only of those candidates approved by the voter. Various rules for determining a winner can be implemented, giving rise to different fairness criteria being satisfied. See Steven J. Brams and M. Remzi Sanver, "Voting Systems That Combine Approval and Preference," in Steven Brams, William Gehrlein and Fred Roberts, eds., *The Mathematics of Preference, Choice and Order: Essays in Honor of Peter C. Fishburn*, Berlin: Springer-Verlag, 2009, 215–237.

16. The Scottish economist Duncan Black (1908–1991) argued that modified simple majority voting is a fair social decision procedure if certain conditions on individual preferences prevented intransitive results from occurring. Examine Black's notions of *single peaked preferences* and *median voter* and his formulation and proofs of some possibility theorems. See Duncan Black, "On the Rationale of Group Decision Making," *Journal of Political Economy* **56** (1948): 23–34; and A. K. Sen and P. K. Pattanaik, "Necessary and Sufficient Conditions for Rational Choice under Majority Decision," *Journal of Economic Theory* **1** (1969): 178–202.

You can find a listing of references and suggestions for additional reading on the book's website, www.wiley.com/college/olinick