OMA 2. izpit 12. 1. 2025

① Za $n \geq 2$ velja $(1-\frac{1}{2})(1-\frac{1}{3})\cdots(1-\frac{1}{n}) = \frac{1}{n}$. (T_n)

T_2 (baza indukcije): $(1-\frac{1}{2}) = \frac{1}{2}$. Drži.

T_n (i.p.): $(1-\frac{1}{2})(1-\frac{1}{2})\cdots(1-\frac{1}{n}) = \frac{1}{n}$.

$T_n \to T_{n+1}$ (i.k.): $(1-\frac{1}{2})(1-\frac{1}{3})\cdots(1-\frac{1}{n})(1-\frac{1}{n+1}) = \frac{1}{n+1}$.

$$(1-\tfrac{1}{2})(1-\tfrac{1}{3})\cdots(1-\tfrac{1}{n})(1-\tfrac{1}{n+1})$$

$$\overset{\text{i.p.}}{=} \frac{1}{n}\left(1-\frac{1}{n+1}\right) = \frac{1}{n}\cdot\frac{n+1-1}{n+1} = \frac{n}{n(n+1)} = \frac{1}{n+1}.$$

② $z^7 + z^5 - 32z^2 - 32 = 0$

$z^5(z^2+1) - 32(z^2+1) = 0$

$(z^5-32)(z^2+1) = 0$

1. $z^2 = -1 \Rightarrow z_{1,2} = \pm i$

2. $z^5 = 32$

$z^5 = 32(\cos 0 + i \sin 0)$

$z_k = 2\left(\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}\right)$, $k = 0, 1, \ldots, 4$.

$z_0 = 2$,

$z_1 = 2\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$

$z_2 = 2\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$

$z_3 = 2\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right)$

$z_4 = 2\left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right)$.

③ $a_n = \dfrac{2^n-1}{2^n+n}$.

a) $a_{n+1} > a_n$

$$\frac{2^{n+1}-1}{2^{n+1}+n+1} - \frac{2^n-1}{2^n+n} > 0$$

$$\frac{(2^{n+1}-1)(2^n+n) - (2^n-1)(2^{n+1}+n+1)}{\quad} > 0$$

$$\frac{\dots}{(2^{n+1}+n+1)-(2^n+n)} > 0$$

$$\frac{\cancel{2}^{2n+1} + \overset{2n2^n \,=}{n2^{n+1}} - 2^n - n - \cancel{2}^{2n+1} - n2^n - 2^n + \overset{2\cdot 2^n \,=}{2^{n+1}} + n + 1}{\underset{2\cdot 2^n =}{2^{n+1}} - 2^n + 1} > 0$$

$$\frac{n2^n+1}{2^n+1} > 0$$ Res, saj sta imenovalec in števec pozitivna.

b) Konvergenca, $\inf a_n$, $\sup a_n$.

Zaporedje je omejeno navzgor:

$$\frac{2^n-1}{2^n+n} < 1 \iff 2^n-1 < 2^n+n \iff n > -1.$$

Ker je zaporedje omejeno navzgor in naraščajoče, je omejeno.

$$\inf a_n = a_1 = \frac{2-1}{2+1} = \frac{1}{3}.$$

$$\sup a_n = \lim_{n\to\infty} \frac{2^n-1}{2^n+n} = 1.$$

4) Konvergenca vrst.

a) $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$.

Primerjalni test. $\frac{1}{n^2+n} < \frac{1}{n^2} \iff n^2 < n^2+n \iff n > 0.$

Vrsta $\sum_{n=1}^{\infty} \frac{1}{n^2}$ konvergira, torej vrsta $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ konvergira.

b) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(3n)!}$

Kvocientni test. $$\lim_{n\to\infty} \frac{\frac{((n+1)!)^2}{(3(n+1))!}}{\frac{(n!)^2}{(3n)!}} = \lim_{n\to\infty} \frac{(n+1)!\,(n+1)!\,\cancel{(3n)!}}{\cancel{(3n+3)!}\,n!\,n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)(n+1)}{(3n+1)(3n+2)(3n+3)} = 0.$$ Torej vrsta konvergira.

c) $\sum_{n=1}^{\infty} \frac{1}{(1+\frac{1}{n})^{n^2}}$.

Korenski test. $$\lim_{n\to\infty} \sqrt[n]{\frac{1}{(1+1/n)^{n^2}}} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} = e^{-1} < 1.$$

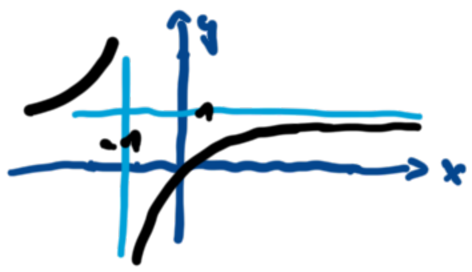
Vrsta konvergira.

5) $f(x) = e^{\frac{x}{x+1}}$

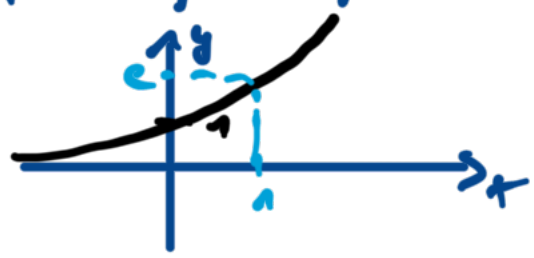
a) Definicijsko območje. $D_f = \mathbb{R} \setminus \{-1\}$.

Zaloga vrednosti.

Graf funkcije $x \mapsto \frac{x}{x+1}$:



Graf funkcije $x \mapsto e^x$



Torej $\frac{x}{x+1}$ doseže vrednosti $I: (-\infty, 1) \cup (1, \infty)$,

$f(x)$ pa doseže vrednosti $e^I = (0, e) \cup (e, \infty)$.

$Z_f = (0, e) \cup (e, \infty)$.

b) Injektivnost.

1. način ($f(x) = f(x') \Rightarrow x = x'$).

$$e^{\frac{x}{x+1}} = e^{\frac{x'}{x'+1}} \Rightarrow \frac{x}{x+1} = \frac{x'}{x'+1} \Rightarrow xx' + x = xx' + x' \Rightarrow x = x'.$$

2. način: Funkciji $x \mapsto \frac{x}{x+1}$ in $x \mapsto e^x$ sta injektivni, kar se vidi iz grafov.

Kompozitum injektivnih funkcij je injektivna funkcija

c) Inverz.

$$x = e^{\frac{y}{y+1}} \quad / \ln$$

$$\ln x = \frac{y}{y+1} \Rightarrow (y+1)\ln x = y \Rightarrow y(\ln x - 1) = -\ln x$$

$$\Rightarrow y = \frac{-\ln x}{\ln x - 1} \Rightarrow f^{-1}(x) = \frac{-\ln x}{\ln x - 1}.$$

5. a) $$\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x^2 + 2x + 4)}{\cancel{(x-2)}(x+2)} = \frac{4+4+4}{4} = 3.$$

b) $$\lim_{x \to 0} \frac{\cos(x^2) - 1}{x^4} = \lim_{x \to 0} \frac{\cos^2(x^2) - 1}{x^4(\cos(x^2) + 1)} = \lim_{x \to 0} \frac{\sin(x^2)}{(x^2)^2 (\cos(x^2) + 1)} = \frac{1}{2}$$

$\to 1$

c) $$\lim_{x \to \infty} (\sqrt{x^2 - 10x + 1} - x) = \lim_{x \to \infty} \frac{x^2 - 10x + 1 - x^2 \quad :x}{\sqrt{x^2 - 10x + 1} + x \quad :x} = \lim_{x \to \infty} \frac{-10 + 1/x}{\sqrt{1 - 10/x + 1/x^2} + 1} = -5$$