1. naloga

$F_0 = 0, \; F_1 = 1, \; F_n = F_{n-1} + F_{n-2}$ za $n \geq 2$

a) $F_{n+1} \leq 2^n$ za $n \geq 0$. (T_n)

Baza indukcije (T_0, T_1)

$n=0$: $F_1 = 1 \leq 2^0$ ✓

$n=1$: $F_2 = 1+0 \leq 2^1$ ✓

Indukcijska predpostavka (T_n)

$F_{n+1} \leq 2^n$

Indukcijski korak $(T_n \wedge T_{n-1} \Rightarrow T_{n+1})$, $n \geq 1$

$\underline{F_{n+2} \leq 2^{n+1}}$

$F_{n+2} = F_{n+1} + F_n \overset{\text{I.P.}}{\leq} 2^n + 2^{n-1} = 2^n\left(1+\frac{1}{2}\right) = \frac{3}{2}2^n \leq 2^{n+1}$.

b) $F_{n-1} \cdot F_{n+1} - F_n^2 = (-1)^n$ za $n \geq 1$

Baza indukcije (T_1, T_2)

$n=1$: $F_0 \cdot F_2 - F_1^2 = (-1)^1$

$0 \cdot 1 - 1^2 = -1$ ✓

$n=2$: $F_1 \cdot F_3 - F_2^2 = (-1)^2$

$1 \cdot 2 - 1^2 = 1$ ✓

Indukcijska predpostavka (T_n)

$F_{n-1} \cdot F_{n+1} - F_n^2 = (-1)^n$

Indukcijski korak $(T_{n-1} \wedge T_n \Rightarrow T_{n+1})$, $n \geq 2$

$\underline{F_n \cdot F_{n+2} - F_{n+1}^2 = (-1)^{n+1}}$

$F_n \cdot F_{n+2} - F_{n+1}^2 = F_n(F_{n+1} + F_n) - F_{n+1}(F_n + F_{n-1})$

$= F_n^2 + \cancel{F_n \cdot F_{n+1}} - \cancel{F_{n+1} \cdot F_n} - F_{n+1} \cdot F_{n-1}$

$= -(F_{n+1} \cdot F_{n-1} - F_n^2) \overset{\text{I.P.}}{=} -(-1)^n = (-1)^{n+1}$.

2. naloga

$\log_6 10$ je iracionalno.

$\log_6 10 = \frac{p}{q}$ (okrajšan ulomek, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $GCD(p,q)=1$).

$6^{p/q} = 10 \quad /^q$

$6^p = 10^q$

$2^p \cdot 3^p = 2^q \cdot 5^q$

$\Rightarrow p = q$ in $3^p = 5^q$

$\Rightarrow 3^p = 5^p$

$\Rightarrow p = q = 0$ (protislovje)

3. naloga

$A = \{x \in \mathbb{R} \mid |x-2| + |x+1| < 9\}$

Rešimo neenačbo

$|x-2| + |x+1| < 9$

i. $\underline{x < -1}$

$-x+2-x-1 < 9$

$x > -4$

$x_1 \in (-4,-1)$

ii. $\underline{-1 \leq x < 2}$

$-x+2+x+1 < 9$

$3 < 9$

$x_2 \in [-1,2)$

iii. $x \geq 2$

$x-2+x+1 < 9$

$x < 5$

$x_3 \in [2,5)$

Skupna rešitev: $x \in (-4,-1) \cup [-1,2) \cup [2,5) = (-4,5)$.

$A = (-4,5)$

$\sup A = 5$, $\inf A = -4$

Minimum in maksimum ne obstajata.

4. naloga

$$z^2 = 2\bar{z} \quad , \; z = a + bi$$

$$(a+bi)^2 = 2(a-bi)$$

$$a^2 + 2abi - b^2 = 2a - 2bi$$

$$\text{Re: } a^2 - b^2 = 2a$$

$$\text{Im: } 2ab = -2b$$

$$2b(a+1) = 0 \quad \Rightarrow \quad b = 0 \quad \text{ali} \quad a = -1$$

1. $b_1 = 0$

$$a^2 = 2a$$

$$a(a-2) = 0 \Rightarrow a_1 = 0 \, , \; a_2 = 2$$

2. $a = -1$

$$1 - b^2 = -2$$

$$b^2 = 3 \quad \Rightarrow \quad b_{3,4} = \pm\sqrt{3}$$

Rešitve: $z_1 = 0, \; z_2 = 2, \; z_3 = -1 + \sqrt{3}i, \; z_4 = -1 - \sqrt{3}i$

5. naloga

$$z^5 = (1+i)^7$$

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i \cdot \sin\frac{\pi}{4}\right)$$

$$(1+i)^7 = \sqrt{2}^7\left(\cos\frac{7\pi}{4} + i \cdot \sin\frac{7\pi}{4}\right) = 2^{7/2}\left(\cos\left(-\frac{\pi}{4}\right) + i \cdot \sin\left(-\frac{\pi}{4}\right)\right)$$

$$z_i = \sqrt[5]{2^{7/2}}\left(\cos\frac{-\frac{\pi}{4} + 2k\pi}{5} + i\sin\frac{-\frac{\pi}{4} + 2k\pi}{5}\right), \quad z \in \{0,1,2,3,4\}$$

$$z_0 = 2^{7/10}\left(\cos\left(-\frac{\pi}{20}\right) + i \cdot \sin\left(-\frac{\pi}{20}\right)\right)$$

$$z_1 = 2^{7/10}\left(\cos\left(\frac{7\pi}{20}\right) + i \cdot \sin\left(\frac{7\pi}{20}\right)\right)$$

$$z_2 = 2^{7/10}\left(\cos\left(\frac{15\pi}{20}\right) + i \cdot \sin\left(\frac{15\pi}{20}\right)\right) = 2^{7/10}\left(-\frac{\sqrt{2}}{2} + i \cdot \frac{\sqrt{2}}{2}\right) = -\sqrt[5]{2} + i\sqrt[5]{2}$$

$$z_3 = 2^{7/10}\left(\cos\left(\frac{23\pi}{20}\right) + i \cdot \sin\left(\frac{23\pi}{20}\right)\right)$$

$$z_4 = 2^{7/10}\left(\cos\left(\frac{31\pi}{20}\right) + i \cdot \sin\left(\frac{31\pi}{20}\right)\right)$$