① $a_n = \frac{n}{n+(\frac{1}{2})^n}$.

a) Zaporedje a_n je naraščajoče in omejeno

Naraščajoče. $a_{n+1} > a_n$

$$\frac{n+1}{n+1+(\frac{1}{2})^{n+1}} > \frac{n}{n+(\frac{1}{2})^n}$$

$$\Rightarrow (n+1)(n+(\tfrac{1}{2})^n) > n(n+1+\tfrac{1}{2}(\tfrac{1}{2})^n)$$

$$\cancel{n^2}+n(\tfrac{1}{2})^n+\cancel{n}+(\tfrac{1}{2})^n > \cancel{n^2}+\cancel{n}+\tfrac{1}{2}n(\tfrac{1}{2})^n$$

$$(n+1)(\tfrac{1}{2})^n > \tfrac{1}{2}n(\tfrac{1}{2})^n \quad /:(\tfrac{1}{2})^n$$

$$n+1 > \tfrac{1}{2}n$$

$$n > -2.$$

Zaporedje je spodaj omejeno z $a_1 = \frac{2}{3}$, saj je naraščajoče

Zaporedje je zgoraj omejeno z 1:

$$\frac{n}{n+(\frac{1}{2})^n} < 1 \Rightarrow n < n+(\tfrac{1}{2})^n \Rightarrow (\tfrac{1}{2})^n > 0.$$

b) limita, ε-okolica za $\varepsilon = 0{,}01$

$$\lim_{n\to\infty} \frac{n}{n+(\frac{1}{2})^n} = 1$$

$$|a_n - 1| < 0{,}01$$

$$-\left(\frac{n}{n+(\frac{1}{2})^n} - 1\right) < 0{,}01 \quad /\cdot(-100)$$

$$\frac{100n}{n+(\frac{1}{2})^n} > 99 \quad /\cdot(n+(\tfrac{1}{2})^n)$$

$$100n > 99n + 99\cdot(\tfrac{1}{2})^n \quad /:99$$

$$\frac{n}{99} > (\tfrac{1}{2})^n \quad /\cdot 99\cdot 2^n$$

$n\cdot 2^n > 99$ velja za $n \geq 5$

Odgovor: $n_0 = 5$.

② Obravnavaj pogojno in absolutno konv. vrst

a) $\sum_{n=1}^{\infty}\left(\frac{4n+1}{2n}\right)^n$

Korenski kriterij: $c_n = \sqrt[n]{a_n} = \frac{4n+1}{2n} > 1$.

Vrsta ne konvergira pogojno in ne konvergira absolutno.

b) $\sum_{n=1}^{\infty}(-1)^n\sqrt{\frac{n+4}{n^2+3}}$

Absolutna konvergenca.

Primerjalni kriterij:

$\sqrt{\frac{n+4}{n^2+3}} > \frac{1}{\sqrt{n}} \quad / \cdot \sqrt{n}\cdot\sqrt{n^2+3}$

$\sqrt{n+4}\cdot\sqrt{n} > \sqrt{n^2+3} \quad /^2$

$n^2+4n > n^2+3$

$4n > 3$

$n > 3/4$ res.

Vrsta ne konvergira absolutno.

Pogojna konvergenca (alternirajoči test)

$a_n = (-1)^n b_n, \quad b_n = \sqrt{\frac{n+4}{n^2+3}}$

1. $b_n \geq 0$ ✓

2. zaporedje je padajoče, $b_{n+1} < b_n$

$\sqrt{\frac{n+5}{(n+1)^2+3}} < \sqrt{\frac{n+4}{n^2+3}} \quad /^2$

$\frac{n+5}{(n+1)^2+3} < \frac{n+4}{n^2+3}$

$(n+5)(n^2+3) < (n+4)((n+1)^2+3)$

$n^3+3n+5n^2+15 < n^3+6n^2+12n+16$

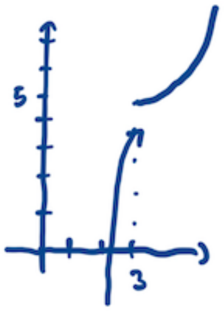
$n^2+9n+1 > 0$ ✓ (Res, saj $n \geq 1$)

3. $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\sqrt{\frac{n+4}{n^2+3}} = \lim_{n\to\infty}\sqrt{\frac{n^2(1/n+4/n^2)}{n^2(1+3/n^2)}} = 0.$

Zaporedje je pogojno konvergentno.

③ $f(x) = \begin{cases} \ln(2x-5)+4, & \text{če } x \leq 3 \quad (y \leq 4) \\ (x-3)^2+5, & \text{če } x > 3 \quad (y > 5) \end{cases}$

a) skica grafa



$Z_f = \left(\frac{5}{2}, \infty\right)$

$D_f = (-\infty, 4] \cup (5, \infty)$

b) Funkcija je injektivna.

Zožitev $f: D_f \to Z_f$ je bijektivna.

Inverz $f_1(x) = \ln(2x-5)+4$

$x = \ln(2y-5)+4$

$e^{x-4} = 2y-5$

$y = \dfrac{e^{x-4}+5}{2}$

Inverz $f_2(x) = (x-3)^2+5$

$x = (y-3)^2+5$

$\pm\sqrt{x-5} = y-3$

$y = \sqrt{x-5}+3$

$$f^{-1}(x) = \begin{cases} \dfrac{e^{x-4}+5}{2}, & \text{če } x \leq 4, \\ \sqrt{x-5}+3, & \text{če } x > 5. \end{cases}$$

4

a) $$\lim_{x \to 1} \frac{\sqrt{1+8x}\ -3}{1-\sqrt{x}} = \lim_{x \to 1} \frac{\sqrt{1+8x}\ -3}{1-\sqrt{x}} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}} \cdot \frac{\sqrt{1+8x}-3}{\sqrt{1+8x}-3}$$

$$= \lim_{x \to 1} \frac{(1+8x-9)(1+\sqrt{x})}{(1-x)(\sqrt{1+8x}-3)} = \lim_{x \to 1} \frac{-(8-8x)\ (1+\sqrt{x})}{(1-x)(\sqrt{1+8x}+3)}$$

$$= -8 \cdot \frac{2}{6} = -\frac{8}{3}$$

b) $$\lim_{x \to 0} \frac{\cos x - 1 + \sin^2 x}{x^2} = \lim_{x \to 0} \left(\frac{\cos x - 1}{x^2} + \frac{\sin^2 x}{x^2} \right)$$

$$= \lim_{x \to 0} \frac{\cos^2 - 1}{x^2(\cos^2 x + 1)} + \lim_{x \to 0} \frac{\sin^2 x}{x^2}$$

$$= \lim \frac{-\sin^2 x}{x^2(\cos^2 x + 1)} + 1$$

$$= -\frac{1}{2} + 1 = \frac{1}{2}$$

⑤ $f(x) = \begin{cases} \frac{|x+1|}{x+1}, & \text{če } x < -1 \\ ax+b, & \text{če } -1 \leq x < 1 \\ \arctan\frac{1}{x-1}, & \text{če } x > 1. \end{cases}$

$$\left.\begin{array}{l} \lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-} \frac{|x+1|}{x+1} = \lim\limits_{x\to -1^-} \frac{-x-1}{x+1} = -1 \\ \lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} (ax+b) = -a+b \end{array}\right\} -a+b=-1$$

$$\left.\begin{array}{l} \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (ax+b) = a+b \\ \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \arctan \overset{\nearrow +\infty}{\frac{1}{x-1}} = \frac{\pi}{2} \end{array}\right\} a+b=\frac{\pi}{2}$$

Rešimo sistem

$$\begin{array}{r} -a+b=-1 \\ a+b=\pi/2 \\ \hline 2b=\frac{\pi}{2}-1 \end{array} \Rightarrow b=\frac{\pi}{4}-\frac{1}{2}$$

$$a=b+1=\frac{\pi}{2}+\frac{1}{2}.$$