Uporaba pravil za odvajanje in enačba tangente na krivuljo

Pravila za odvajanje:

$$\begin{aligned}(f+g)' &= f'+g'\\(f-g)' &= f'-g'\\(f\cdot g)' &= f'\cdot g'\\\left(\frac{f}{g}\right)' &= \frac{f'}{g'}\end{aligned}$$

Enačba tangente v $T(x_0, y_0)$:

$$y - y_0 = f'(x_0)(x - x_0)$$

1. Izračunaj odvod naslednjih funkcij:
   (a) $f(x) = x^2 + 4x + 3$
   (b) $f(x) = -x^2 + 4x + 5$
   (c) $f(x) = 2x^3 + 6x^2 + 6x + 2$
   (d) $f(x) = \dfrac{x+1}{x-1}$
   (e) $f(x) = \dfrac{2x+4}{x-3}$
   (f) $f(x) = \dfrac{x^2+4x+3}{x^2-9}$
   (g) $f(x) = \dfrac{x^2-7x+6}{x-2}$.
   (h) $f(x) = \dfrac{3}{x^2}$.
2. Določi enačbo tangente na graf funkcije f :
   (a) $f(x) = x^2 + 4x + 3$ v točki $T(-1, y)$.
   (b) $f(x) = -x^2 + 4x + 5$ v točki $T(1, y)$.
   (c) $f(x) = 2x^3 + 6x^2 + 6x + 2$ v točki $T(-2, y)$
   (d) $f(x) = \dfrac{x+1}{x-1}$ v točki $T(2, y)$.
   (e) $f(x) = \dfrac{2x+4}{x-3}$ v točki $T(2, y)$
   (f) $f(x) = \dfrac{x^2+4x+3}{x^2-9}$ v točki $T(4, y)$
3. Vsako funkcijo in njeno tangento nariši.
4. Na krivuljo
   $$f(x) = \frac{2x+1}{x-2}$$
   položi tangento, ki gra skozi koordinatno izhodišče. Določi tangento.
5. V kateri točki krivulje
   $$f(x) = x^2 - 6x - 7$$
   je tangenta vzporedna z abscisno osjo?
6. V kateri točki krivulje
   $$f(x) = x^2 - 2x + 5$$
   je tangenta vzporedna s premico $y = -x$?
7. Upoštevaj da je hitrost odvod poti po času t:
   $$v(t) = s'(t)$$
   in izračunaj hitrost, če je
   (a) $s = \frac{a}{2}t^2$
   (b) $s = s_0 + v_0 t$
   (c) $s = v_0 t + \frac{a}{2}t^2$
   (d) $s = \frac{t^3}{3} - 2t^2 + 3t$

   Kdaj je hitrost v zadnjem primeru enaka 0?
8. Naj bodo a, b, c in d realne konstante. Dokaži, da je funkcija
   $$f(x) = \frac{ax+b}{cx+d}$$
   strogo monotona na definicijskem območju.

Kot med krivuljami

Kot med krivuljama: …je kot med tangentama na krivulji v presečišču: $y = k_1x + n_1, y = k_2x + n_2,$

$$\text{tg } \phi = \left|\frac{k_1 - k_2}{1 + k_1k_2}\right|$$

$k_1 = k_2 \Rightarrow$ tangenti sta vzporedni$\Rightarrow \phi = 0°$
$k_1 \cdot k_2 = -1 \Rightarrow$ tangenti sta pravokotni$\Rightarrow \phi = 90°$

9. Izračunaj kot med krivuljama

   (a)
   $$y = 7x - 1$$
   $$y = 3x + 3$$

   (b)
   $$y = x^{-2}$$
   $$y = \frac{x^2 - 1}{2}$$

   (c)
   $$y = x^2 + 1$$
   $$y = -x + 7$$

   (d)
   $$y = \frac{2}{1 + x^2}$$
   $$y = 3x^2 - 2$$

   (e) *
   $$y = x^3 - 3x^2 + 3x - 1$$
   $$y = x^2 - 2x + 1$$

10. Določi lokalne ekstreme fukcij s pomočjo odvoda:

   (a) $f(x) = x^2 + 7x + 6$
   (b) $f(x) = -2x^2 - 8x - 6$
   (c) $f(x) = x^3 + 4x^2 - 11x - 30$
   (d) q

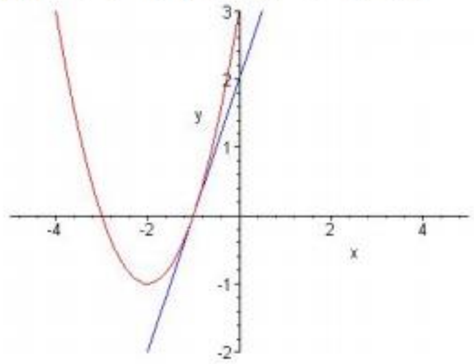
Rešitve

1. (a) $f'(x) = 2x + 4$
   (b) $f'(x) = -2x + 4$
   (c) $f'(x) = 6x^2 + 12x + 6$
   (d) $f'(x) = -\dfrac{2}{(x-1)^2}$
   (e) $f'(x) = -\dfrac{10}{(x-3)^2}$
   (f) $f'(x) = -\dfrac{4}{(x-3)^2}$
   (g) $f'(x) = \dfrac{x^2 - 4x + 8}{(x-2)^2}$
   (h) $f'(x) = -6x^{-3}$

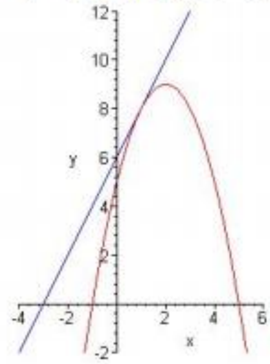
2. Enačbe tangent so:
   (a) $T(-1, 0), x_0 = 1, y_0 = 0,$
   $k = f'(x_0) = f'(-1) = 2$
   $\rightsquigarrow \quad y = 2x + 2$
   (b) $T(1, 8), x_0 = 1, y_0 = 8,$
   $k = f'(x_0) = f'(1) = 2$
   $\rightsquigarrow \quad y = 2x + 6$
   (c) $T(-2, -2), x_0 = -2, y_0 = -2,$
   $k = f'(x_0) = f'(-2) = 6$
   $\rightsquigarrow \quad y = 6x + 10$
   (d) $T(2, 3), x_0 = 2, y_0 = 3,$
   $k = f'(x_0) = f'(2) = -2$
   $\rightsquigarrow \quad y = -2x + 7$
   (e) $T(2, -8), x_0 = 2, y_0 = -8,$
   $k = f'(x_0) = f'(2) = -10$
   $\rightsquigarrow \quad y = -10x + 12$
   (f) $T(4, 5), x_0 = 4, y_0 = 5,$
   $k = f'(x_0) = f'(4) = -4$
   $\rightsquigarrow \quad y = -4x + 21$

3. Tangente in grafi v sliki:

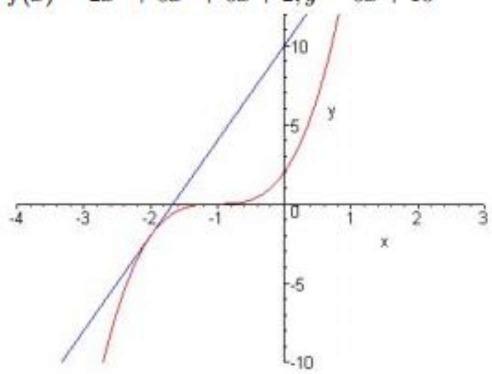
(a) $f(x) = x^2 + 4x + 3, y = 2x + 2$



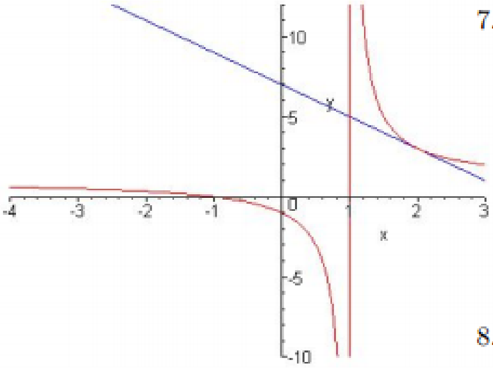
(b) $f(x) = -x^2 + 4x + 5, y = 2x + 6$



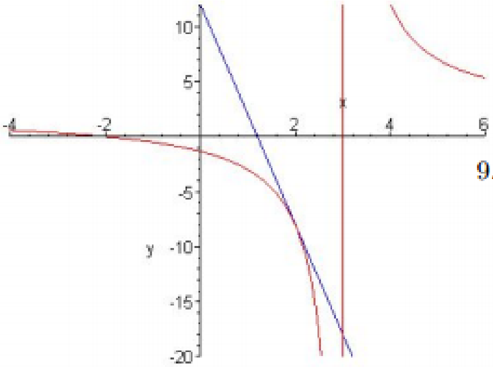
(c) $f(x) = 2x^3 + 6x^2 + 6x + 2, y = 6x + 10$



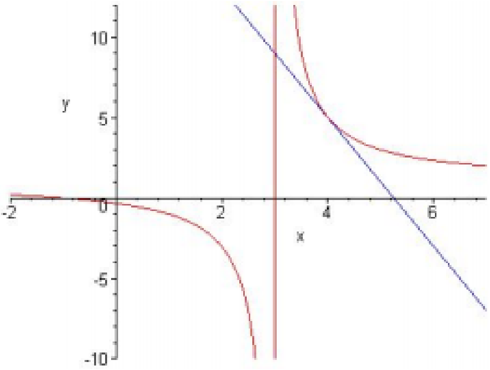
(d) $f(x) = \dfrac{x+1}{x-1}, y = -2x + 7$



(e) $f(x) = \frac{2x+4}{x-3}, y = -10x + 12$



(f) $f(x) = \frac{x^2+4x+3}{x^2-9}, y = -4x + 21$



4.

5. $T(3, -16)$

6. $T(\frac{1}{2}, \frac{17}{4})$

7. $s'(t) = v(t)$

$$\begin{aligned} s'(t) &= at \\ s'(t) &= v_0 \\ s'(t) &= v_0 + at \\ s'(t) &= t^2 - 4t + 3 = (t-3)(t-1) \end{aligned}$$

Hitrost $v = 0$, če je $t = 3$ ali $t = 1$.

8. Odvod funkcije je

$$f'(x) = \frac{bc - ad}{(cx+d)^2},$$

kar pomeni, da ne spremeni predznak na definicijskem območju. Imenovalec je namreč pozitiven, števec pa je konstanta.

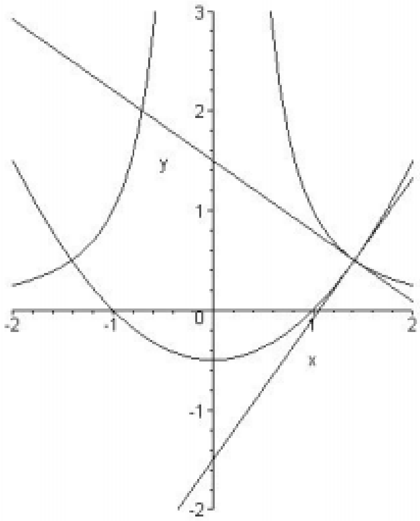
9. Koti med krivuljama:

Zadostuje poznati smerna koeficienta tangent v skupnih točkah:

(a) $k_1 = 7, k_2 = 3,$
$\text{tg}\,\phi = \left|\frac{7-3}{1+3\cdot 7}\right| = \frac{2}{11}, \phi =$

(b) Izračunamo skupne točke:
$\frac{1}{x^2} = \frac{x^2-1}{2},$
$x^4 - x^2 - 2 = 0$
$(x^2-2)(x^2+1) = 0$
$x_{1,2} = \pm\sqrt{2}, y_{1,2} = \frac{1}{2}.$
$P_1(\sqrt{2}, \frac{1}{2}).P_2(-\sqrt{2}, \frac{1}{2})$
Določimo smerne koeficiente tangent:
$f(x) = \frac{1}{x^2} \Rightarrow f'(x) = -\frac{2}{x^3},$
$g(x) = \frac{x^2-1}{2} \Rightarrow g'(x) = x,$
(i):
$k_1 = f'(x_0) = f'(\sqrt{2}) = -\frac{\sqrt{2}}{2}$
$k_2 = g'(x_0) = f'(\sqrt{2}) = \sqrt{2}$
$k_1 \cdot k_2 = -1 \Rightarrow \phi = 90°$
(ii):

$k_1 = f'(x_0) = f'(-\sqrt{2}) = \frac{\sqrt{2}}{2}$
$k_2 = g'(x_0) = f'(-\sqrt{2}) = -\sqrt{2}$
$k_1 \cdot k_2 = -1 \Rightarrow \phi = 90°$



Prikazani sta tangenti v $(\sqrt{2}, \frac{1}{2})$,
$y = -\frac{\sqrt{2}}{2}x + \frac{3}{2}, y = \sqrt{2}x - \frac{3}{2}$, zaradi sodosti obeh funkcij to velja tudi za drugo presečišče

(c) Določimo skupne točke:
$x^2 + 1 = -x + 7$
$x^2 + x - 6 = 0$
$(x-2)(x+3) = 0 \ x_1 = -3, x_2 = 2$
$P_1(-3, f(-3)) = P_1(-3, 10)$
$P_2(2, f(2)) = P_2(2, 5)$
$f(x) = x^2 + 1, f'(x) = 2x$
(i):
$k_1 = -1, k_2 = f'(x_1) = f'(-3) = -6$

$$\text{tg } \phi = \left|\frac{-1-(-6)}{1+(-1)(-6)}\right| = \frac{5}{7}$$

(ii):
$k_1 = -1, k_2 = f'(x_2) = f'(2) = 4$

$$\text{tg } \phi = \left|\frac{-1-4}{1+(-1)4}\right| = \frac{5}{3}$$

(d) $\frac{2}{1+x^2} = 3x^2 - 2$
$3x^4 + x^2 - 4 = 0$
Z vpeljavo nove spremenljivke $t = x^2$ rešimo enačbo
$3t^2 + t - 4 = 0, \quad t_1 = 1, t_2 < 0$
$x_{1,2} = \pm 1, y_{1,2} = 1$
(i): $f(x) = 2/(1+x^2), g(x) = 3x^2 - 2$
$f'(x) = \frac{-4x}{(1+x^2)^2}, g'(x) = 6x$
$k_1 = f'(x_1) = f'(1) = 6, k_2 = g'(x_1) = g'(1) = -1$

$$\text{tg } \phi = \left|\frac{6-(-1)}{1+(-1)(6)}\right| = \frac{7}{5}$$

(ii):
$k_1 = f'(x_2) = f'(-1) = -6, k_2 = g'(x_2) = g'(-1) = 1$

$$\text{tg } \phi = \left|\frac{-6-(+1)}{1+(1)(-6)}\right| = \frac{7}{5}$$

(e) $x^3 - 3x^2 + 3x - 1 = x^2 - 2x + 1$
$x^3 - 2x^2 + 5x - 2 = 0$
S Hornerjevim algoritmom poiščemo ničle: $x_1 = 1(2\times), x_2 = 2$.
Določimo odvoda:
$f(x) = x^3 - 3x^2 + 3x - 1, f'(x) = 3x^2 - 6x + 3$
$g(x) = x^2 - 2x + 1, g'(x) = 2x - 2$
(i):
$k_1 = f'(x_1) = f'(1) = 0$
$k_2 = g'(x_1) = g'(1) = 0$
Ker je $k_1 = k_2$, je

$$\phi = 0°.$$

(ii):
$k_1 = f'(x_2) = f'(2) = 3$
$k_2 = g'(x_2) = g'(2) = 2$

$$\text{tg } \phi = \left|\frac{3-2}{1+3\cdot 2}\right| = \frac{1}{7}$$